A universal algebraic approach to definability in categories

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Two traditions

Universal Algebra	**Categorical Logic**
operations	connectives
terms	natural transformations
equations	internal logic
definability	P$\mathcal{V}$Ds

This talk builds a bridge between these two viewpoints.

The motivating problem

In a topos, the subobject functor

$$\mathrm{Sub}_{\mathcal{E}} : \mathcal{E}^{op} \to \textbf{Heyt}$$

encodes the internal logical structure of $\mathcal{E}$.

Internal logical operations are natural transformations

$$\alpha : \mathrm{Sub}_{\mathcal{E}}^{n} \to \mathrm{Sub}_{\mathcal{E}}.$$

Examples:

- conjunction,
- implication,
- Lawvere–Tierney topologies.

The motivating problem

Main Question

Can definability theory be developed inside the internal logic of a category?

Subquestion

How can term representability and equational definability be expressed for internal connectives?

Abstracting the situation

We isolate the algebraic fragment of categorical logic.

A propositional $\mathcal{V}$ doctrine (P$\mathcal{V}$D) is a functor

$$F : \mathcal{C}^{op} \to \mathcal{V},$$

where:

- $\mathcal{C}$ has finite products,
- $\mathcal{V}$ is a variety of algebras.

Interpretation:

- $F(A)$: predicates/formulas on A,
- reindexing: substitution,
- algebraic operations: internal connectives.

External algebraic semantics

Given a P$\mathcal{V}$D

$$F : \mathcal{C}^{op} \to \mathcal{V},$$

we consider the associated class of algebras

$$\mathcal{V}_F = \{F(A) \mid A \in \mathcal{C}\}.$$

The P$\mathcal{V}$D F determines an external universal-algebraic semantics.

Internal connectives

Let $F : \mathcal{C}^{op} \to \mathcal{V}$ be a P$\mathcal{V}$D and $n \in \mathbb{N}$. An n-ary F-connective is a natural transformation
$$\alpha : F^n \Rightarrow F.$$

$$\begin{tikzcd} F(A)^n \arrow[r, "\alpha_A"] \arrow[d, "F(f)^n"'] & F(A) \arrow[d, "F(f)"] \\ F(B)^n \arrow[r, "\alpha_B"'] & F(B) \end{tikzcd}$$

Naturality imposes strong algebraic coherence.

Example: Lawvere–Tierney topologies

Let $\mathcal{E}$ be a topos.

The associated subobject functor

$$\mathrm{Sub}_{\mathcal{E}} : \mathcal{E}^{op} \to \mathsf{Heyt}$$

is a propositional Heyt doctrine.

A Lawvere–Tierney topology on $\mathcal{E}$ is a morphism

$$j : \Omega_{\mathcal{E}} \to \Omega_{\mathcal{E}}$$

Example: Lawvere–Tierney topologies

satisfying:

1. $x \leq j(x)$,
2. $j(j(x)) = j(x)$,
3. $j(x \wedge y) = j(x) \wedge j(y)$.

By the Yoneda lemma,

$$\mathcal{E}(-, \Omega_{\mathcal{E}}) \cong \mathrm{Sub}_{\mathcal{E}}(-),$$

so j induces a natural transformation

$$j : \mathrm{Sub}_{\mathcal{E}} \to \mathrm{Sub}_{\mathcal{E}}.$$

Hence, Lawvere–Tierney topologies are unary $\mathrm{Sub}_{\mathcal{E}}$-connectives satisfying closure-like algebraic conditions.

Term representability

Let $F : \mathcal{C}^{op} \to \mathcal{V}$ be a P$\mathcal{V}$D and let

$$\alpha : F^n \Rightarrow F$$

be an n-ary F-connective.

Definition

We say that α is $\mathcal{L}_{\mathcal{V}}$-*term representable* if there exists an n-ary $\mathcal{L}_{\mathcal{V}}$-term $t(x_1, \ldots, x_n)$ such that

$$\alpha_A(a_1, \ldots, a_n) = t^{F(A)}(a_1, \ldots, a_n)$$

for every $A \in \mathcal{C}$ and every $a_1, \ldots, a_n \in F(A)$.

Equivalently, α is induced pointwise by the same algebraic term in every algebra $F(A)$.

Equational definability

Let $F : \mathcal{C}^{op} \to \mathcal{V}$ be a P$\mathcal{V}$D and let

$$\alpha : F^n \Rightarrow F$$

be an n-ary F-connective.

Definition

We say that α is $\mathcal{L}_{\mathcal{V}}$-*equationally definable* iff there exist $n+1$-ary $\mathcal{L}_{\mathcal{V}}$-terms

$$p_1, \ldots, p_k, q_1, \ldots, q_k$$

such that

$$\mathcal{V}_F \models \alpha(\vec{x}) = y \leftrightarrow \bigwedge_{i=1}^{k} p_i(\vec{x}, y) = q_i(\vec{x}, y)$$

for all $1 \leq i \leq k$.

Representable P$\mathcal{V}$Ds

Assume the P$\mathcal{V}$D

$$F : \mathcal{C}^{op} \to \mathcal{V}$$

is represented by an object $R \in \mathcal{C}$. I.e., there exists a natural isomorphism

$$\iota : \mathcal{O}_{\mathcal{V}} F \Rightarrow \mathcal{C}(-, R).$$

where $\mathcal{O}_{\mathcal{V}} : \mathcal{V} \to \mathsf{Set}$ is the forgetful functor. That is,

$$\begin{tikzcd}
\mathcal{O}_{\mathcal{V}} F(A) \arrow[r, "\iota_A"] \arrow[d, "\mathcal{O}_{\mathcal{V}} F(f)"'] & \mathcal{C}(A, R) \arrow[d, "- \circ f"] \\
\mathcal{O}_{\mathcal{V}} F(B) \arrow[r, "\iota_B"'] & \mathcal{C}(B, R)
\end{tikzcd}$$

Representable P$\mathcal{V}$Ds

Theorem: (Representable P$\mathcal{V}$Ds as internal algebras)

Let $F : \mathcal{C}^{op} \to \mathcal{V}$ be a representable P$\mathcal{V}$D represented by an object $R \in \mathcal{C}$. Then R carries the structure of an internal $\mathcal{V}$-algebra in $\mathcal{C}$. Conversely, if $R \in \mathcal{C}$ is an internal $\mathcal{V}$-algebra, then the representable functor

$$\mathcal{C}(-, R) : \mathcal{C}^{op} \to \mathbf{Set}$$

canonically lifts to a P$\mathcal{V}$D

$$F_R : \mathcal{C}^{op} \to \mathcal{V}$$

such that

$$\mathcal{O}_{\mathcal{V}}(F_R(A)) = \mathcal{C}(A, R)$$

for every $A \in \mathcal{C}$.

Representable P$\mathcal{V}$Ds: Term representability

Let $F : \mathcal{C}^{op} \to \mathcal{V}$ be a representable P$\mathcal{V}$D represented by $R \in \mathcal{C}$, and let

$$\alpha : F^n \Rightarrow F$$

be an n-ary F-connective.

Recall the canonical element

$$\vec{u}_{R^n} := (\iota_{R^n}^{-1}(\pi_1), \ldots, \iota_{R^n}^{-1}(\pi_n)) \in F(R^n)^n.$$

Theorem

The connective α is $\mathcal{L}_{\mathcal{V}}$-term representable iff

$$\alpha_{R^n}(\vec{u}_{R^n}) \in \mathrm{Sg}^{F(R^n)}(\iota_{R^n}^{-1}(\pi_1), \ldots, \iota_{R^n}^{-1}(\pi_n)).$$

Equivalently, the canonical value of α belongs to every projection-generated subalgebra of $F(R^n)$.

Representable P$\mathcal{V}$Ds: Equational definability

Let $F : \mathcal{C}^{op} \to \mathcal{V}$ be a representable P$\mathcal{V}$D represented by $R \in \mathcal{C}$, and let

$$\alpha : F^n \Rightarrow F$$

be an n-ary F-connective.

Define the extended canonical tuple

$$\vec{u}_{R^{n+1}} := (\iota^{-1}_{R^{n+1}}(\pi_1), \dots, \iota^{-1}_{R^{n+1}}(\pi_{n+1})) \in F(R^{n+1})^{n+1}.$$

Theorem

The connective α is $\mathcal{L}_{\mathcal{V}}$-equationally definable iff there exist $(n+1)$-ary $\mathcal{L}_{\mathcal{V}}$-terms $p_1, \dots, p_k, q_1, \dots, q_k$ such that

$$\alpha_{R^{n+1}}(u_1, \dots, u_n) = u_{n+1} \iff p_i^{F(R^{n+1})}(\vec{u}_{R^{n+1}}) = q_i^{F(R^{n+1})}(\vec{u}_{R^{n+1}})$$

for every $1 \leq i \leq k$.

A non-term-representable connective

The category Grph of directed graphs is a presheaf topos

$$\mathsf{Grph} \simeq \widehat{\mathsf{C}},$$

hence it carries a subobject classifier Ω.

The object Ω is the graph

$$\begin{tikzcd} 0 \arrow[loop left, "0"] \arrow[rr, bend left, "b"] & & 1 \arrow[ll, bend left, "a"] \arrow[loop right, "c"] \arrow[loop right, "1"] \end{tikzcd}$$

Consider the subgraph

$$\begin{tikzcd} 1 \arrow[loop right, "c"] \end{tikzcd}$$

which induces a unary connective

$$B : \Omega \to \Omega.$$

A non-term-representable connective

Observation

The connective B satisfies

$$B(\top) \neq \top.$$

Hence B cannot be term definable (as term operations preserve constants).

Big picture

Internal logic $\Longrightarrow$ P$\mathcal{V}$Ds

Representability $\Longrightarrow$ Internal $\mathcal{V}$-algebras

Internal connectives $\Longrightarrow$ Definability theory

Core idea

Global categorical questions about internal connectives reduce to local algebraic questions about canonical elements in $F(R^n)$.

Conclusion

This work proposes a universal algebraic approach to definability phenomena arising in categorical logic.

Starting from a P$\mathcal{V}$D

$$F : \mathcal{C}^{op} \to \mathcal{V},$$

internal logical operations become natural transformations

$$\alpha : F^n \Rightarrow F,$$

and representability allows these connectives to be analyzed through the algebraic structure of a single representing object.

Conclusion

When F is representable, its representing object R canonically becomes an internal $\mathcal{V}$-algebra in $\mathcal{C}$, allowing methods from algebraic definability theory to be extended to internal categorical settings.

The main results show that:

- Term representability reduces to algebraic closure conditions,
- Equational definability reduces to finite algebraic descriptions,
- Global categorical behavior can be studied locally in $F(R^n)$ or $F(R^{n+1})$ through the canonical elements $\vec{u}_{R^n}$ or $\vec{u}_{R^{n+1}}$, respectively.

In particular, this opens the possibility of systematically classifying internal modalities and geometric connectives using methods from universal algebra.

Thank you !

Scan for a summary of the talk !