Depth-bounded Intuitionistic Propositional Logic

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Background problem

Let $\mathbf{L}$ be a logical system with consequence relation $\vdash_{\mathbf{L}}$. In many cases:

- $\vdash_{\mathbf{L}}$ is undecidable or intractable;
- proof-search exhibits combinatorial explosion even when the search space is constrained (e.g. subformula properties).

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The standard presentation of a logical system abstracts away from resource limitations and presupposes idealized agents.

Need for an approximation discipline that accounts for inferential depth.

The depth parameter

The depth hypothesis

Given any idealized consequence relation $\vdash$, there exists an intuitively meaningful parameter d ranging over a partially ordered set $(P, \leq)$ such that $\vdash$ can be approximated by subrelations $\vdash_d$, where

$$\vdash_d \subseteq \vdash_{d'} \quad \text{whenever } d \leq d' \tag{1}$$

$$\bigcup_{d \in P} \vdash_d = \vdash \quad \text{(in the limit)} \tag{2}$$

$$\vdash_d \text{ is tractable for any fixed } d \tag{3}$$

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The parameter d must correspond to some robust measure of inferential depth, not merely the number of steps in some proof procedure.

*See (D'Agostino, Gabbay, Larese, & Modgil, 2024).

Idealized intuitionistic information states

- In the standard relational semantics of **IPL**, information states and the associated forcing relation are idealized.

$$s \Vdash B \text{ for all } B \text{ in } \Gamma, \text{ and } \Gamma \vdash_{\mathbf{IPL}} A, \text{ then } s \Vdash A.$$

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- But $\vdash_{\mathbf{IPL}}$ is PSPACE-complete (Statman, 1979).
- So **IPL** is marred by the logical omniscience phenomenon.
- We address this by constructing the standard forcing relation as the ideal limit of a hierarchy of depth-bounded forcing relations $\Vdash_d$, $d \in \mathbb{N}$, measuring the available inferential depth.

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- This may include:
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  - trusted external verifications and procedures.
- In the context of intuitionistic logic, assent is conclusive: once justified, it is not retracted under future informational growth.

Three informational values

Value	**Informal meaning**
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- U may arise for different reasons:
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- Different informational situations may provide certificates for compound formulas even when no certificate is available for their constituents.

Non-deterministic tables: disjunction and conjunction

$\widetilde{\vee}$	T	F	U
T	T	T	T
F	T	F	U
U	T	U	T , U

When no certificate is available for either disjunct, the current informational situation may or may not contain a certificate for the disjunction.

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$\widetilde{\wedge}$	T	F	U
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When no certificate is available for either conjunct, the current informational situation may or may not contain a certificate that the conjunction is not verifiable.

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When no certificate is available for either conjunct, the current informational situation may or may not contain a certificate that the conjunction is not verifiable.

The tables are informationally dual. However, since $v(A) = \mathsf{F}$ is not equivalent to $v(\neg A) = \mathsf{T}$, neither connective is defined in terms of the other and negation.

Non-locality of implication

- The current informational situation is not always sufficient to determine whether a conditional is non-verifiable.
- To certify $v(A \to B) = \mathsf{F}$ in general, we must consider possible refinements of the current informational situation.

Non-locality of implication

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- To certify $v(A \to B) = \mathsf{F}$ in general, we must consider possible refinements of the current informational situation.
- $\to$ requires a genuinely global semantic clause.
- A certificate for $A \to B$ consists in a theoretical procedure that transforms any certificate of verification for A into a certificate of verification for B.

The non-deterministic table for implication

$\widetilde{\to}$	T	F	U
T	T	F	F , U
F	T	T , F , U	T , F , U
U	T	F , U	T , F , U

- Red occurrences of F indicate cases in which non-verifiability cannot be determined locally.
- In all such cases, and only in them, an additional global condition must be checked.
- $v_x(A \to B) = \mathrm{F} \iff \exists y \geq x, v_y(A) = \mathrm{T}$ and $v_y(B) = \mathrm{F}$.

*Different from those in (Leme, Coniglio & Lopes, 2025) for full **IPL**.

0-depth **IPL**-structures

Definition

A 0-depth **IPL**-structure, $\mathcal{M}$, is a tuple $(S, \leq, \{v_x\}_{x \in S})$, where

- S is a non-empty set of informational situations,
- $\leq$ is a pre-order on S, and
- v_x is a mapping $\mathcal{F} \mapsto \{\mathsf{T}, \mathsf{F}, \mathsf{U}\}$ satisfying:
  - $v_x(A \circ B) \in \widetilde{\circ}(v_x(A), v_x(B))$, where $\circ$ is $\wedge$, $\vee$ or $\rightarrow$;
  - and the global conditions:

    (1) $v_x(A) = \mathsf{T}$ and $x \leq y \Longrightarrow v_y(A) = \mathsf{T}$;

    (2) $v_x(A \rightarrow B) = \mathsf{F} \iff \exists y \geq x, v_y(A) = \mathsf{T}$ and $v_y(B) = \mathsf{F}$.

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$\Gamma \models_0 A$ iff $\forall \mathcal{M}, \forall x \in S,\ v_x(\Gamma) = \mathsf{T} \Longrightarrow v_x(A) = \mathsf{T}$.

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Virtual space and refinements

- Virtual information: information introduced hypothetically in order to explore possible informational completions.
- It may be introduced in order to derive consequences not available at depth 0.
- Depth-bounded forcing is defined relative to a finite set Δ of formulas: $v_x : \Delta \to \{T, F, U\}$.
- Virtual space: a designated set of formulas $\mathcal{V}(\Delta)$, whose members may be used as virtual information. Typically, $\mathcal{V}(\Delta) = \mathrm{sub}(\Delta)$.

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Definition

A refinement v'_x of v_x, $\quad v_x \sqsubseteq v'_x$,

- extends v_x: whenever $v_x(A) \in \{\mathsf{T}, \mathsf{F}\}$, $v'_x(A) = v_x(A)$;
- v'_x may assign T or F to some formulas in $\mathcal{V}(\Delta)$ that were previously U.

Depth-bounded forcing and consequence

Definition

- $x \Vdash_0^{\mathsf{T}} A$ iff $v_x(A) = \mathsf{T}$, $\quad x \Vdash_0^{\mathsf{F}} A$ iff $v_x(A) = \mathsf{F}$.
- For $d > 0$,

  $x \Vdash_d^{\mathsf{T}} A$ iff $\exists B \in \mathcal{V}(\Delta)$ such that, $\forall v_x' \sqsupseteq v_x$, with $v_x'(B) \in \{\mathsf{T}, \mathsf{F}\}$, $\quad x' \Vdash_{d-1}^{\mathsf{T}} A$.

*Similarly for $\Vdash_d^{\mathsf{F}}$.

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- $x \Vdash_0^{\mathsf{T}} A \quad \text{iff} \quad v_x(A) = \mathsf{T}\,, \qquad x \Vdash_0^{\mathsf{F}} A \quad \text{iff} \quad v_x(A) = \mathsf{F}\,.$
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Definition

Let $\Delta = \Gamma \cup \{A\}$.

$$\Gamma \models_d A \text{ iff } \forall \mathcal{M},\ \forall x \in S,\ x \Vdash_d^{\mathsf{T}} \Gamma \Longrightarrow x \Vdash_d^{\mathsf{T}} A.$$

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$$\Gamma \models_d A \text{ iff } \forall \mathcal{M}, \forall x \in S, x \Vdash_d^{\mathsf{T}} \Gamma \Longrightarrow x \Vdash_d^{\mathsf{T}} A.$$

The forcing relations are computed relative to $\Delta = \Gamma \cup \{A\}$ and to the corresponding virtual space $\mathcal{V}(\Delta)$.

Semantic intelim trees (symmetry of $\vee$ and $\wedge$ is assumed)

$$\frac{\mathsf{T}\,A : x}{\mathsf{T}\,A \vee B : x} \qquad \frac{\begin{array}{l}\mathsf{F}\,A : x\\ \mathsf{F}\,B : x\end{array}}{\mathsf{F}\,A \vee B : x} \qquad \frac{\mathsf{F}\,A : x}{\mathsf{F}\,A \wedge B : x} \qquad \frac{\begin{array}{l}\mathsf{T}\,A : x\\ \mathsf{T}\,B : y\\ \boxed{x \leq z \text{ and } y \leq z}\end{array}}{\mathsf{T}\,A \wedge B : z}$$

$$\frac{\mathsf{T}\,B : x}{\mathsf{T}\,A \to B : x} \qquad \frac{\begin{array}{l}\mathsf{T}\,A : x\\ \mathsf{F}\,B : y\\ \boxed{x \leq y}\end{array}}{\mathsf{F}\,A \to B : x} \qquad \frac{\begin{array}{l}\mathsf{T}\,A : x\\ \boxed{x \leq y}\end{array}}{\mathsf{F}\,\neg A : y} \qquad \frac{\mathsf{F}\,A \vee B : x}{\begin{array}{l}\mathsf{F}\,A : x\\ \mathsf{F}\,B : x\end{array}}$$

$$\frac{\mathsf{T}\,A \wedge B : x}{\begin{array}{l}\mathsf{T}\,A : x\\ \mathsf{T}\,B : x\end{array}} \qquad \frac{\begin{array}{l}\mathsf{T}\,A \vee B : x\\ \mathsf{F}\,A : y\\ \boxed{x \leq y}\end{array}}{\mathsf{T}\,B : x} \qquad \frac{\begin{array}{l}\mathsf{F}\,A \wedge B : y\\ \mathsf{T}\,A : x\\ \boxed{x \leq y}\end{array}}{\mathsf{F}\,B : y} \qquad \frac{\begin{array}{l}\mathsf{T}\,A \to B : x\\ \mathsf{T}\,A : y\\ \boxed{x \leq z \text{ and } y \leq z}\end{array}}{\mathsf{T}\,B : z}$$

$$\frac{\begin{array}{l}\mathsf{T}\,A \to B : x\\ \mathsf{F}\,B : y\\ \boxed{x \leq y}\end{array}}{\mathsf{F}\,A : y} \qquad \frac{\mathsf{F}\,A \to B : x}{\begin{array}{l}\mathsf{T}\,A : y\\ \mathsf{F}\,B : y\\ x \leq y\end{array}}\; y \text{ new} \qquad \frac{\begin{array}{l}\mathsf{T}\,\neg A : x\\ \boxed{x \leq y}\end{array}}{\mathsf{F}\,A : y} \qquad \frac{\mathsf{F}\,\neg A : x}{\begin{array}{l}\mathsf{T}\,A : y\\ x \leq y\end{array}}\; y \text{ new}$$

$$\frac{\begin{array}{l}\mathsf{T}\,A : x\\ \mathsf{F}\,A : y\\ \boxed{x \leq y}\end{array}}{\times} \qquad \frac{}{\mathsf{T}\,A : x \;\mid\; \mathsf{F}\,A : x}$$

*See (D'Agostino & D. Gabbay, 1994) and (Solares-Rojas, Baldi, & Rodriguez, 2026).

Final remarks

Proposition

The unbounded system is complete.

Claim

Every depth-bounded approximation is tractable.

This paves the way to extend the approach to a variety of logics that admit of relational semantics.

References

M. D'Agostino & D. Gabbay, 1994. A generalization of analytic deduction via labelled deductive systems. Part I: Basic substructural logics. *Journal of Automated Reasoning*, 13(2):243–281.

M. D'Agostino, D. Gabbay, C. Larese, & S. Modgil, 2024. *Depth-bounded Reasoning. Volume 1: Classical Propositional Logic*. College Publications.

R. Leme, M. Coniglio & B. Lopes, 2025. A new decision method for Intuitionistic Logic by 3-valued non-deterministic truth-tables. *Journal of Symbolic Logic*, 1-36.

A. Solares-Rojas, P. Baldi, & R. O. Rodriguez, 2026. Free-variable KE tableaux for IPL. Manuscript accepted for publication in *Journal of Logic and Computation*.

Statman, R., 1979. Intuitionistic propositional logic is polynomial-space complete. *Theoretical Computer Science*, 9(1): 67-72.

¡Muchas gracias!