Axiomatizing the logics of finite Gödel Kripke models

R. O. Rodriguez

Joint work with Amanda Vidal

SLALM XXI. Bogotá, Jun 1-5, 2026

Institute of Computer Science of Buenos Aires University-CONICET

The authors was supported by MOSAIC project 101007627.

Table of contents

Introduction

Modal Many Valued Logics

In approximate reasoning it is usual to deal with assertions like to *"John is possibly tall"*, with the intended meaning that the fuzzy (many valued) proposition *"John is tall"* is "*possible to be true*".

Technically, we need then to combine elements of many-valued logics (to model fuzziness) and of modal logics (to model the notion of uncertainty).

$\Rightarrow$ **modal many-valued logics**

Here, we are going to focus on **modal Gödel logic**.

i.e. the modal extension of Gödel logic:

Heyting Logic + $(\varphi \to \psi) \vee (\psi \to \varphi)$

$$e(\varphi \wedge \psi) := \min(e(\varphi), e(\psi))$$

$$e(\varphi \to \psi) := \begin{cases} 1 & \text{if } e(\varphi) \leq e(\psi), \\ e(\psi) & \text{otherwise} \end{cases}$$

Some definitions

Definition

An (standard valued) **Gödel Kripke model** $\mathfrak{M}$ is a structure $\langle W, R, e\rangle$ with W a non-empty set, $R\colon W^2 \to [0,1]$ and $e\colon W \times \mathcal{V} \to [0,1]$, with e being point-wise a Gödel homomorphism and furthermore,

$$e(v, \Diamond\varphi) := \bigvee_{w\in W} R(v,w) \wedge e(w,\varphi),\; e(v,\Box\varphi) := \bigwedge_{w\in W} R(v,w) \to e(w,\varphi)$$

A model is **crisp** whenever $R\colon W^2 \to \{0,1\}$, i.e., $R \subseteq W^2$. (we write Rvw instead of $\langle v, w\rangle \in R$).

- MG is the logic of all Gödel Kripke models;
- KG is the logic of all crisp Gödel Kripke models;
- $\mathcal{L}_\Box/\mathcal{L}_\Diamond$ refer to the corresponding mono-modal fragment of $\mathcal{L}$.
- $MG_\Box$ and $MG_\Diamond$ are the logics corresponding to the mentioned mono-modal fragments. The same for $KG_\Box$ and $KG_\Diamond$.

Known axiomatizations

$$MG_\Box = KG_\Box$$

$K_\Box$: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$

$Z_\Box$: $\neg\neg\Box\varphi \to \Box\neg\neg\varphi$

$N_\Box$: $\vdash \varphi$ implies $\vdash \Box\varphi$

$$MG_\Diamond$$

$K_\Diamond$: $\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$

$Z_\Diamond$: $\Diamond\neg\neg\varphi \to \neg\neg\Diamond\varphi$

$F_\Diamond$: $\neg\Diamond\bot$

$N_\Diamond$: $\vdash \varphi \to \psi$ implies $\vdash \Diamond\varphi \to \Diamond\psi$.

The system $KG_\Diamond$ results by replacing in the axiomatization of $MG_\Diamond$ rule $N_\Diamond$ by the following one:

$$N^c_\Diamond : \vdash (\varphi \to \psi) \vee \chi \textit{ implies } \vdash (\Diamond\varphi \to \Diamond\psi) \vee \Diamond\chi.$$

The system MG (i.e., the bi-modal godel logic) results from adding to the union of $MG_\Box$ and $MG_\Diamond$, Fischer Servi's connecting axioms:

$$FS_1 : \Diamond(\varphi \to \psi) \to (\Box\varphi \to \Diamond\psi), \qquad FS_2 : (\Diamond\varphi \to \Box\psi) \to \Box(\varphi \to \psi).$$

Finally, the system KG is obtained by adding to MG the so-called Dunn's axiom:

$$Cr : \Box(\varphi \vee \psi) \to (\Box\varphi \vee \Diamond\psi).$$

More Definitions (witnessing conditions)

A model $\mathfrak{M}$ is **witnessed** if for every formula φ,

$$e(v, \Box\varphi) = R(v, w) \rightarrow e(w, \varphi) \text{ for some } w \in W,$$
$$e(v, \Diamond\varphi) = R(v, w) \wedge e(w, \varphi) \text{ for some } w \in W.$$

For crisp models: $e(v, \Box\varphi) = e(w, \varphi)$ for some Rvw, and $e(v, \Diamond\varphi) = e(u, \varphi)$ for some Rvu.

- MG^{ω}, KG^{ω} are the logics of all corresponding witnessed models;

Partially witnessed models: $\mathfrak{M}$ is:
* **$\Box$-witnessed or $\Diamond$-witnessed** whenever the witnessing conditions hold for the modal formulas starting with the corresponding modality.
* **quasi-witnessed** whenever it is $\Diamond$-witnessed and for each φ, v, either $e(v, \Box\varphi) = 0$ or it is witnessed.
* $\Diamond$-$\top$ **witnessed** whenever for each φ, v, if $e(v, \Diamond\varphi) = 1$ then there is some w with $R(v, w) = 1$ and $e(w, \varphi) = 1$.

Known results & open problems

Many of the above-mentioned logics have been axiomatized [2,3,5,6].

- $MG_{\Diamond}$ is witnessed [2]. $KG_{\Box}$ and $KG_{\Diamond}$ are decidable [5];
- $KG, MG, (MG/KG)_{\Box}$ are not witnessed (counter-example [2] arises from $\Box\neg\neg p \to \neg\neg\Box p$),
- $(MG/KG)_{\Box} + \Box\neg\neg p \to \neg\neg\Box p$ is complete with respect to 0-$\Box$-witnessed models (from [p. 198][2]), i.e. those models where $e(v, \Box\varphi) = 0$ implies the existence of u such that $e(u, \varphi) = 0 < R(v, u)$.
- $KG_{\Diamond}$ is not witnessed, (counter-example [5] arises from $(\Diamond p \to \Diamond q) \to ((\neg\Diamond q \vee \Diamond(p \to q)))$

Known results & open problems

Many of the above-mentioned logics have been axiomatized [2,3,5,6].

- $MG_\Diamond$ is witnessed [2]. $KG_\Box$ and $KG_\Diamond$ are decidable [5];
- $KG, MG, (MG/KG)_\Box$ are not witnessed (counter-example [2] arises from $\Box\neg\neg p \to \neg\neg\Box p$),
- $(MG/KG)_\Box + \Box\neg\neg x \to \neg\neg\Box x$ is complete with respect to 0-$\Box$-witnessed models.
- $KG_\Diamond$ is not witnessed, (counter-example [5] arises from $(\Diamond p \to \Diamond q) \to ((\neg\Diamond q \vee \Diamond(p \to q)))$

Open questions:

- **Does** $(MG/KG)_\Box + \Box\neg\neg x \to \neg\neg\Box x$ **satisfy the FMP?**
- **Can we get finite axiomatizations of the witnessed companions of the previous logics, and of** $KG_\Diamond$**? (in [4], this is attempted)** From [5], is $KG_\Diamond + (\Diamond p \to \Diamond q) \to ((\neg\Diamond q \vee \Diamond(p \to q))$ witnessed?
- **Do the above logics enjoy some nice partial witnessed completeness?**

Failure of witnessed completeness & partial witnessed conditions

On the failure of witnessed completeness ($\Box$ case)

Lemma

$KG_\Box + \Box\neg\neg x \to \neg\neg\Box x$, and $MG_\Box + \Box\neg\neg x \to \neg\neg\Box x$ are not complete with respect to witnessed models.

Proof: We can prove that the formula

$$(\Box(((x \to y) \to y) \wedge (y \to x)) \to ((\Box x \to \Box y) \to \Box y)$$

is sound for (valued) witnessed models, but is not a consequence of $KG_\Box + \Box\neg\neg x \to \neg\neg\Box x$.

Hence, the same happens for KG and MG.

On the failure of witnessed completeness ($\Box$ case)-example

For proving that

$$(\Box(((x \to y) \to y) \wedge (y \to x)) \to ((\Box x \to \Box y) \to \Box y) \ (*)$$

is fasified on a model of $KG_\Box + \Box\neg\neg x \to \neg\neg\Box x$, we consider the following one: $W = u \cup \{v_i \colon i \in \omega, i > 1\}$, $Ruv_i = 1$ for all i, and $e(v_i, x) = \frac{1}{2} + \frac{1}{i}$, $e(v_i, y) = \frac{1}{2}$ (the values at u are irrelevant). It is clear that $e(v_i, x) > e(v_i, y)$ for any i so $e(v_i, (((x \to y) \to y) \wedge (y \to x)) = 1$ for any i, and therefore, $e(u, \Box((((x \to y) \to y) \wedge (y \to x)) = 1$.
On the other hand, since $e(u, \Box x) = e(u, \Box y) = \frac{1}{2}$, we get that $e(u, (\Box x \to \Box y) \to \Box y) = \frac{1}{2}$, proving that the formula (*) is not valid in the model.

The fact that this model is $KG_\Box^\star$ model follows from the fact that it is 0-$\Box$-witnessed. Since $KG_\Box^\star$ is complete with respect to 0-$\Box$-witnessed models [p. 198][2], this concludes the proof.

On the failure of witnessed completeness ($\Diamond$ case)

Lemma

$KG_\Diamond + (\Diamond x \to \Diamond y) \to (\neg\Diamond y \vee \Diamond(x \to y))$ is not complete w.r.t. witnessed models.

In the same spirit as before the formula

$$(\Diamond p \to \Diamond q) \wedge ((\Diamond q \to \Diamond z) \to \Diamond z)) \to (\Diamond z \vee \Diamond((p \to q) \wedge ((q \to z) \to z))$$

is sound for (valued) witnessed models, but is not a consequence of $KG_\Diamond + (\Diamond x \to \Diamond y) \to (\neg\Diamond y \vee \Diamond(x \to y))$.

Partial witnessed conditions

Lemma

KG (and $KG_\Diamond$) is complete with respect to $\Diamond$-$\top$ witnessed models.

Proof: Over the canonical model from [4], for $v, \Diamond\varphi$ with $v(\Diamond\varphi) = 1$, there must exist w with Rvw and $w(\chi) < w(\varphi)$ for each $v(\Diamond\chi) < 1$.

$$w'(p) \coloneqq \begin{cases} w(p) & \text{if } w(p) < w(\varphi), \\ 1 & \text{otherwise} \end{cases}$$

We can prove this characterization extends to formulas, i.e., $w'(\xi) = w(\xi)$ if $e(\xi) < w(\varphi)$, and 1 otherwise.
It is easy to check that Rvw', and since $w'(\varphi) = 1$, this witnesses $\Diamond\varphi$ at the top.
Remark: The previous result implies that if the minimal bi-modal Gödel logic is extended with the axiom $\Diamond\top$, the frame satisfying the property of seriality i.e. $\forall v \exists u : R(v, u) = 1$. It was an open problem pointed out in [Remark pag. 49][3].

Partial witnessed conditions

Lemma

MG is complete with respect to $\Diamond$-witnessed models.

Proof: Working over the canonical model from [3], for v, $\Diamond\varphi$ with $e(v, \Diamond\varphi) = \alpha > 0$, it holds that

$$\varphi, \{\xi\colon \Box\xi \in \Omega, v(\Box\xi) \geq \alpha\} \not\vdash_{MG} \bigvee\{\theta\colon \Diamond\theta \in \Omega, v(\Diamond\theta) < \alpha\}$$

This implies the existence of $u \in W$ with $u(\xi) = u(\varphi) = 1$ for each such ξ, and $u(\theta) < \alpha$, implying:

$$u(\theta_1) < u(\theta_2) \text{ for any } v(\Diamond\theta_1) < v(\Box\theta_2), v(\Diamond\theta_1) < \alpha \tag{1}$$

$$u(\theta_1) \leq u(\theta_2) \text{ for any } v(\Diamond\theta_1) \leq v(\Box\theta_2), v(\Diamond\theta_1) < \alpha \tag{2}$$

$$u(\theta) = 0 \Longrightarrow v(\Box\theta) = 0, \qquad v(\Diamond\theta) = 0 \Longrightarrow u(\theta) = 0 \tag{3}$$

Using the above, we can define an endomorphism g, such that $u' = g \circ u \in W$ and that $u'(\varphi) \to v(\Diamond\varphi) = 1 \to \alpha = \alpha$ and

Axiomatizing the logics of finite models

Axiomatizing the logics of finite models

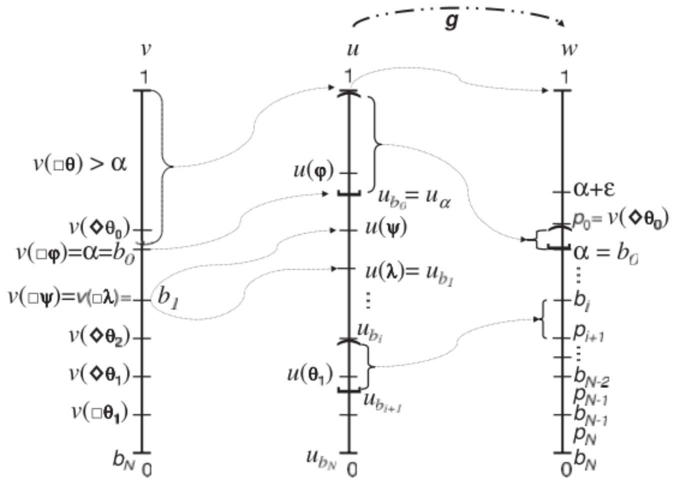
$$
\begin{aligned}
W_\Box :=& (\Box(((x \to y) \to y) \wedge (y \to x)) \to ((\Box x \to \Box y) \to \Box y) \\
W_\Diamond :=& ((\Diamond p \to \Diamond q) \wedge ((\Diamond q \to \Diamond z) \to \Diamond z)) \to \\
& (\Diamond z \vee \Diamond((p \to q) \wedge ((q \to z) \to z))
\end{aligned}
$$

Theorem

$MG + W_\Box$ is complete with respect to witnessed models.

The proof uses the fact that MG is complete with respect to $\Diamond$-witnessed models, and an almost identical proof to the next theorem.

Axiomatizing the logics of finite models



$S^F vw > (\alpha + \varepsilon)$

Axiomatizing the logics of finite models

Theorem

$KG + W_\Box + W_\Diamond$ is complete with respect to witnessed models.

Proof: Working over the canonical model from [6], let h and $\Box\varphi$. Let $1 > \alpha := h(\Box\varphi)$ and $\beta := max\{h(\Box\psi) \colon h(\Box\psi) < \alpha\}$. [Recall here and elsewhere, we quantify over a finite set of formulas]

$\chi := \bigwedge\{\psi \colon h(\Box\psi) > \alpha\} \qquad \phi := \bigwedge\{\psi \colon h(\Box\psi) = \alpha\}$.

$h(\Box(\chi \to ((\varphi \to \phi) \to \phi) \wedge (\phi \to \varphi))) < 1$

This gives a world g with Rhg, $g(\phi) \geq g(\varphi) > \alpha$ and $g(\chi) > g(\varphi)$. Let

$$\sigma(x) := \begin{cases} x & \text{if } x > e(g,\varphi), \\ \frac{x-\beta}{g(\varphi)-\beta}(\alpha-\beta)+\beta & \text{if } x \in (\beta, e(g,\varphi)], \\ x & \text{otherwise.} \end{cases}$$

$g' = \sigma \circ g \in W$, and we can prove that Rhg'. Since $g'(\varphi) = \alpha$, this witnesses $\Box\varphi$.

Axiomatizing the logics of finite models ($\Diamond$ proof)

Let h and $\Diamond\varphi$, and let $0 < \alpha \coloneqq h(\Diamond\varphi) < 1$ and
$\beta \coloneqq min\{h(\Diamond\psi) \colon h(\Diamond\psi) > \alpha\}$.
$\chi \coloneqq \bigvee\{\psi \colon h(\Diamond\psi) < \alpha\} \qquad \phi \coloneqq \bigvee\{\psi \colon h(\Diamond\psi) = \alpha\}$.

Instantiating $W_\Diamond$ and using that $\Diamond$ distributes over $\vee$, we get that
$h(\Diamond\chi \vee \Diamond((\phi \to \varphi) \wedge ((\varphi \to \chi) \to \chi))) = 1$

Since $h(\Diamond\chi) < \alpha$, necessarily $h(\Diamond((\phi \to \varphi) \wedge ((\varphi \to \chi) \to \chi)))) = 1$.
Since we already proved that the model is $\top$-witnessed there is $g \in W$ with Rhg and $g((\phi \to \varphi) \wedge ((\varphi \to \chi) \to \chi)) = 1$, therefore,
$g(\phi) \leq g(\varphi) < \alpha$ and $g(\chi) < g(\varphi)$ (since $g(\chi) < 1$) necessarily).

$$\sigma(x) \coloneqq \begin{cases} x & \text{if } x < e(g,\varphi), \\ \frac{\beta - x}{\beta - e(g,\varphi)}(\beta - \alpha) + \alpha & \text{if } x \in [e(g,\varphi), \beta), \\ x & \text{otherwise.} \end{cases}$$

$g' = \sigma \circ g \in W$, and we can prove that Rhg'. Since $g'(\varphi) = \alpha$, this witnesses $\Diamond\varphi$.

Open questions & References

Ongoing/future questions

- The above logics coincide with the ones of all finite Gödel chains, and those of $[0,1]_{\uparrow} = \{0, 1/2, 2/3, 3/4, ..., 1\}$. Can this work give us inspiration on how to axiomatize different modal Gödel logics, in the spirit of [Baaz,Preining,Zach, "First-Order Gödel Logics"], e.g.,
  - modal logics over $[0,1]_{\downarrow} = \{1, 1/2, 1/3, 1/4, ..., 0\}$;
  - modal logics over a Gödel set with a finite set of downward/upward accumulation points (in a given order)?
- Is the global logic arising from $\mathcal{MG}^w$ axiomatizable?
- We would like to consider modal extensions when the accesibility relation satisfies special properties like reflexivity, transitivity, symmetry, and etc.

THANK YOU!

References

[1] X. Caicedo, G. Metcalfe, R. O. Rodriguez and J. Rogger (2017) **Decidability of order-based modal logics.** *Journal of Computer and System Sciences*, 88:53-74.

[2] X. Caicedo and R. O. Rodriguez (2010) **Standard Gödel modal logics.***Studia Logica*, 94(2):189–214.

[3] X. Caicedo and R. O. Rodriguez (2015) **Bi-modal Gödel logic over [0, 1]-valued Kripke frames.***Journal of Logic and Computation*, 25(1):37–55.

[4] M. Ferrari, C. Fiorentini and R. O. Rodriguez (2025) **A Gödel Modal Logic over Witnessed Crisp Models**. *Proceedings of TABLEAUX 2025*. LNCS, vol 15980.

[5] G. Metcalfe and N. Olivetti (2011) **Towards a proof theory of Gödel modal logics.** *Log. Methods Comput. Sci.* 7(2), 1–27

[6] R. O. Rodriguez and A. Vidal (2021) **Axiomatization of Crisp Gödel Modal Logic.** *Studia Logica*, 109, 367–395.