Semilattice Semantics for a Relevance Logic with Constructive Negation

Alejandro Pinilla Barrera
Juan Carlos Agudelo Agudelo

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UNIVERSIDAD DE ANTIOQUIA
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Introduction

The main concern in logic is valid deductions, that is, the consequence relation between premises and conclusions. From the classical perspective, there is no requirement of a meaningful connection between them. This conception has led to several “paradoxes”:

- $A \rightarrow (B \rightarrow A)$,
- $\sim A \rightarrow (A \rightarrow B)$,
- $(A \wedge \sim A) \rightarrow B$,
- $A \rightarrow (B \rightarrow B)$,
- $A \rightarrow (B \vee \sim B)$.

Anderson and Belnap (1958) initiated a new research program, building on the works of Lewis (1912, 1917), Ackermann (1956) and Church (1951), which ultimately gave rise to relevance logic.

Introduction

The central idea of this program is that, in a valid argument, the premises must be relevant to the conclusion. Two main approaches support this idea:

- **The principle of variable sharing:** a necessary (though not sufficient) condition for an implication from A to B is that A and B share relevant content, that is, a variable.
- **The method of tracking premises in a derivation:** track the use of hypothesis in a derivation in order to ensure that they are actually used to the conclusion.

Introduction

Anderson (1963) explicitly formulated a set of open problems concerning his and Belnap's system R. Three were particularly fundamental:

- The admissibility of Ackermann's rule γ.
- The provision of an adequate semantics.
- The decision problem.

The semantic problem has been especially challenging because, as Urquhart (1972) observes, any semantic proposal for relevance logics must address the difficulties posed by negation, whose behavior in these systems often conflicts with core relevance principles.

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Natural Deduction Formulation

The central idea of this approach is to permit the introduction of temporary hypotheses, together with a device that facilitates the bookkeeping of their use and discharge.

In the case of relevance systems, several authors, such as Anderson and Belnap Jr (1975), employ the subscripting device.

Remark 1: Subscripting Device

The **subscripting device** consists in assigning a distinct numeral to each introduced hypothesis and propagating this numeral through each application of an inference rule, thereby tracking which hypotheses are used. When a hypothesis is discharged, the corresponding subscript is removed.

Natural Deduction Formulation

Definition 1: Subscripted Formula

A **subscripted formula** of a logic L is an expression of the form A_α, where A is a formula of L and $\alpha \in \mathcal{P}(\mathbb{N})$ represents the set of undischarged assumptions on which A depends.

Definition 2: Derivation Rules for Relevant Implication

$$\dfrac{\begin{array}{c} A_{\{k\}} \\ \vdots \\ B_{\alpha \cup \{k\}} \end{array}}{(A \to B)_\alpha}\ [\to I] \qquad \dfrac{(A \to B)_\alpha \quad A_\beta}{B_{\alpha \cup \beta}}\ [\to E]$$

Remark 2

This system is known as the system of relevant implication, denoted by $\mathsf{R}_\to$.

Natural Deduction Formulation with Subscripts

Definition 3: Derivation in Natural Deduction Systems

Let A_α be a subscripted formula of of a logic L. A **derivation of A_α from a set of subscripted formulas** $\Gamma = \{B_{1_{\{k_1\}}}, \ldots, B_{n_{\{k_n\}}}\}$ of L is a tree constructed with the rules of L, where A_α is the root and Γ is the set of undischarged assumptions, i.e., the elements $B_{i_{\{k_i\}}}$ being leaves and $\alpha = \{k_1, \ldots, k_n\}$.

Definition 4: Derivability

Let A_α be a subscripted formula of a logic L. A_α is **derivable** in L from a set of subscripted formulas $\Gamma = \{B_{1_{\{k_1\}}}, \ldots, B_{n_{\{k_n\}}}\}$ of L, denoted by $\Gamma \vdash_{\mathsf{L}} A_\alpha$, if there is a derivation of A_α from Γ.

Natural Deduction Formulation

We link the notion of derivability between subscripted and unsubscripted formulas by means of the following definition.

Definition 5

A formula A of a logic L is **derivable** in L from a set of formulas $\Gamma = \{B_1, \ldots, B_n\}$ of L, denoted by $\Gamma \vdash_{\mathsf{L}} A$, if for some $\{k_1, \ldots, k_n\} \in \mathcal{P}(\mathbb{N})$, it follows that $B_{1_{\{k_1\}}}, \ldots, B_{n_{\{k_n\}}} \vdash_{\mathsf{L}} A_\alpha$, where $\alpha = \{k_1, \ldots, k_n\}$.

Remark 3

A formula obtained with the empty set as subscript is called a theorem, since it does not depend on any hypotheses.

Semilattice Semantics

The semilattice semantics proposed by Urquhart (1972) provides a model for interpreting logical systems by means of the structure of a join-semilattice with identity.

Remark 4

The intuitive idea behind the semilattice semantics is that the elements of the semilattice represent **pieces of information**, that is, basic statements about the subject under discussion.

Example 1

- In physics, *pieces of information* may correspond to statements about experimental results.
- In mathematics, they may represent basic facts about numbers.

Semilattice Semantics

Given a family S of pieces of information about a given subject, a natural question then arises: what can be said about the structure of S?

- Given any two pieces of information x and y, we can combine them to form a new piece of information $x \sqcup y$, which contains all the information in x together with all the information in y.
- The identity element 0 corresponds to the empty piece of information.

Remark 5

In this semantics, a valuation function assigns to each propositional variable p a set of pieces of information: those that support the truth of p, or for which the truth of p can be accepted. This valuation is extended inductively to all formulas.

Semilattice Semantics

Definition 6: Join Semilattice

A **join-semilattice** is a structure $\langle S, \sqcup \rangle$, where S is a set and $\sqcup$ is a binary operation on S, called the **join**, satisfying the following identities for all $x, y, z \in S$:

$$x \sqcup x = x \qquad \text{(Idempotency)},$$
$$(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z) \qquad \text{(Associativity)},$$
$$x \sqcup y = y \sqcup x \qquad \text{(Commutativity)}.$$

Remark 6

An **identity** of a join-semilattice $\langle S, \sqcup \rangle$ is an element $0 \in S$ such that $0 \sqcup x = x$ for all $x \in S$. When a join-semilattice $\langle S, \sqcup \rangle$ has an identity element 0, it is denoted by $\langle S, \sqcup, 0 \rangle$.

Semilattice Semantics

Definition 7: Semilattice Frame

A **semilattice frame** is a join-semilattice with identity.

Definition 8: Semilattice Model

A **semilattice model** for a logic L is a structure $\langle S, \sqcup, 0, v\rangle$, where $\langle S, \sqcup, 0\rangle$ is a semilattice frame and $v : V \to \mathcal{P}(S)$ is a valuation function.

Definition 9: Verification Relation

Given a semilattice model $\mathcal{M} = \langle S, \sqcup, 0, v\rangle$ for $R_{\to}$, a **verification relation**, denoted by $\models_{\mathcal{M}}$, between elements of S and formulas of $R_{\to}$ is inductively defined by:

$$x \models_{\mathcal{M}} p \text{ iff } x \in v(p), \text{ for all } p \in V,$$
$$x \models_{\mathcal{M}} A \to B \text{ iff for all } y \in S, \text{ whenever } y \models_{\mathcal{M}} A \text{ then } x \sqcup y \models_{\mathcal{M}} B.$$

Semilattice Semantics

Definition 10

A formula A of $\mathrm{R}_{\to}$ is **verified in a semilattice model** $\mathcal{M} = \langle S, \sqcup, 0, v\rangle$ for $\mathrm{R}_{\to}$ if $0 \models_{\mathcal{M}} A$.

Definition 11

A formula A of $\mathrm{R}_{\to}$ is **valid in the semilattice semantics for** $\mathrm{R}_{\to}$, denoted by $\models_{\mathrm{R}_{\to}} A$, if A is verified in every semilattice model for $\mathrm{R}_{\to}$.

Theorem 1: Weak Adequacy Theorem for the Semilattice Semantics for $\mathrm{R}_{\to}$

Let $A \in \mathcal{F}_{\mathrm{R}_{\to}}$. Then $\vdash_{\mathrm{R}_{\to}} A$ if and only if $\models_{\mathrm{R}_{\to}} A$.

Proof. See Urquhart (1971).

Semilattice Semantics

To establish the strong version of the adequacy theorem, we generalize the notions of verification and validity.

Definition 12

Let $\Gamma = \{B_1, \ldots, B_n\}$ be a set of formulas of $\mathrm{R}_\to$ and A be a formula of $\mathrm{R}_\to$. We say that A is a **consequence of Γ in a semilattice model** $\mathcal{M} = \langle S, \sqcup, 0, v\rangle$ for $\mathrm{R}_\to$, denoted by $\Gamma \models_{\mathcal{M}} A$, if for all $x_1, \ldots, x_n \in S$, whenever $x_1 \models_{\mathcal{M}} B_1, \ldots, x_n \models_{\mathcal{M}} B_n$, it follows that $x_1 \sqcup \cdots \sqcup x_n \models_{\mathcal{M}} A$.

Definition 13

Let $\Gamma = \{B_1, \ldots, B_n\}$ be a set of formulas of $\mathrm{R}_\to$ and A be a formula of $\mathrm{R}_\to$. We say that A is a **consequence of Γ in the semilattice semantics for** $\mathrm{R}_\to$, denoted by $\Gamma \models_{\mathrm{R}_\to} A$, if $\Gamma \models_{\mathcal{M}} A$ holds in every semilattice model $\mathcal{M} = \langle S, \sqcup, 0, v\rangle$ for $\mathrm{R}_\to$.

Semilattice Semantics

Theorem 2: Strong Adequacy Theorem for the Semilattice Semantics for $\mathsf{R}_{\rightarrow}$

Let $\Gamma = \{A_1, \ldots, A_n\}$ be a set of formulas of $\mathsf{R}_{\rightarrow}$ and B be a formula of $\mathsf{R}_{\rightarrow}$. Then $\Gamma \vdash_{\mathsf{R}_{\rightarrow}} B$ if and only if $\Gamma \models_{\mathsf{R}_{\rightarrow}} B$.

Proof. See Urquhart (1972).

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Syntax of SR$^+$: Conjunction

The intuitive introduction and elimination rules for conjunction are:

$$\frac{A_\alpha \qquad B_\beta}{(A \wedge B)_{\alpha\cup\beta}}\ [\wedge I] \qquad\qquad \frac{(A \wedge B)_\alpha}{A_\alpha}\ [\wedge E_1] \quad \frac{(A \wedge B)_\alpha}{B_\alpha}\ [\wedge E_2]$$

However, these rules lead to irrelevance as it allows us to prove the paradox $A \to (B \to A)$. Introduction and elimination rules for conjunction that preserve relevance can be found in the following definition.

Definition 14: Derivation Rules for Conjunction

$$\frac{A_\alpha \qquad B_\alpha}{(A \wedge B)_\alpha}\ [\wedge I] \qquad\qquad \frac{(A \wedge B)_\alpha}{A_\alpha}\ [\wedge E_1] \quad \frac{(A \wedge B)_\alpha}{B_\alpha}\ [\wedge E_2]$$

Syntax of SR$^+$: Disjunction

A similar case happens with disjunction. The more intuitive introduction and elimination rules for disjunction are:

$$\frac{A_\alpha}{(A \vee B)_\alpha}\ [\vee I_1] \qquad \frac{B_\alpha}{(A \vee B)_\alpha}\ [\vee I_2] \qquad\qquad \frac{(A \vee B)_\alpha \qquad \begin{matrix} A_{\{h\}} \\ \vdots \\ C_{\beta \cup \{h\}} \end{matrix} \qquad \begin{matrix} B_{\{l\}} \\ \vdots \\ C_{\delta \cup \{l\}} \end{matrix}}{C_{\alpha \cup \beta \cup \delta}}\ [\vee E]$$

However, these rules also make the paradox $A \to (B \to A)$ a theorem.

Syntax of SR^+: Disjunction

To preserve relevance, correct introduction and elimination rules for disjunction can be found in the following definition.

Definition 15: Derivation Rules for Disjunction

$$\frac{A_\alpha}{(A \vee B)_\alpha}\,[\vee I_1] \qquad \frac{B_\alpha}{(A \vee B)_\alpha}\,[\vee I_2] \qquad \frac{(A \vee B)_\alpha \quad \begin{matrix} A_\alpha \\ \vdots \\ C_{\beta\cup\alpha} \end{matrix} \quad \begin{matrix} B_\alpha \\ \vdots \\ C_{\beta\cup\alpha} \end{matrix}}{C_{\alpha\cup\beta}}\,[\vee E]$$

Remark 7

The system with the introduction and elimination rules for implication, conjunction and disjunction in Definitions 2, 14, and 15 is referred as SR^+, and it was first introduced in Charlwood (1978). This system is equivalent to the natural deduction system without subscripts introduced by Prawitz (1965).

Semantics of SR^+

By extending the semilattice semantics with the following clauses

$$x \models_{\mathcal{M}} A \wedge B \text{ iff } x \models_{\mathcal{M}} A \text{ and } x \models_{\mathcal{M}} B,$$
$$x \models_{\mathcal{M}} A \vee B \text{ iff } x \models_{\mathcal{M}} A \text{ or } x \models_{\mathcal{M}} B,$$

and previous semantical definitions (Definitions 8, 9, 10, and 11), we can establish a weak adequacy theorem for SR^+.

Theorem 3: Weak Adequacy Theorem for the Semilattice Semantics for SR^+

Let A be a formula of SR^+. Then $\vdash_{\mathsf{SR}^+} A$ if and only if $\models_{\mathsf{SR}^+} A$.

Proof. See Charlwood (1978).

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The Problem of Negation in Relevance Logic

According to Copeland (1979), negation is among the most controversial notions in relevance logic because relevance logicians aim to develop a consequence relation that avoids the paradoxes of material implication.

Retaining a fully classical negation within a relevant system makes it difficult to block principles such as *Ex Falso Quodlibet*. This tension gives rise to a fundamental dilemma:

- If contradictions are nowhere true, it becomes unclear how relevant logics can adequately model inconsistent information without collapsing into triviality.
- If contradictions are allowed to be true in some contexts, negation can no longer behave in a fully classical manner.
- Rejecting principles such as $A \to (B \vee \sim B)$ requires admitting situations in which bivalence fails.

The Problem of Negation in Relevance Logic

Remark 8

Negation should be designed to preserve the central relevance-theoretic requirement, while blocking negative paradoxes (such as $A \to (\sim A \to B)$) and the principle of explosion (Mares, 2004).

The most influential relevant system is the one presented axiomatically by Anderson and Belnap Jr (1975): R. In this system negation is introduced with the following principles:

$$(A \to \sim A) \to \sim A \qquad \text{(Reductio)}$$

$$(A \to \sim B) \to (B \to \sim A) \qquad \text{(Contraposition)}$$

$$\sim\sim A \to A \qquad \text{(Double Negation)}$$

Alternatively, negation can be defined in the style of Johansson (1937) by setting $\sim A := A \to f$, where f is a falsity constant. Under this definition, reductio and contraposition become theorems, whereas double negation must still be postulated as an axiom.

Negation in Relevance Logic: Relational Semantics

The most influential account of negation is provided by the relational semantics of Routley and Routley (1972), who introduce an operation on what they call *set-ups* (interpreted as *situations*), also known as the *Routley star device*. The truth condition for negation is given by:

$$x \models_{\mathcal{M}} \sim A \quad \text{iff} \quad x^* \not\models_{\mathcal{M}} A.$$

This condition allows for both inconsistencies and failures of bivalence, and the validation of several negation principles such as the double negation laws.

However, this approach has been the subject of considerable interpretative criticism.

Negation in Relevance Logic: Semilattice Semantics

In the semilattice semantics the problem of introducing negation remains unresolved. If we follow the classical approach, we can extend the semilattice semantics wit the following clause:

$$x \models_{\mathcal{M}} \sim A \quad \text{iff} \quad x \not\models_{\mathcal{M}} A,$$

but, it validates $(A \wedge \sim A) \to B$.

Urquhart (1972) proposes extending the semilattice semantics with a function C under which the semilattice S is closed, satisfying $CCx = x$ and $C0 = 0$. Negation is then defined by:

$$x \models_{\mathcal{M}} \sim A \quad \text{iff} \quad Cx \not\models_{\mathcal{M}} A.$$

This form of negation validates double negation and De Morgan's laws, and invalidates the paradoxical formulas and disjunctive syllogism. However, it fails to validate reductio and contraposition.

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The SR System: Introduction

The notion of constructive negation originates in the work of David Nelson (Nelson, 1949) who introduced a constructive negation as part of his system of constructive logic. Unlike the intuitionistic treatment of negation as a derived connective defined by:

$$\sim A := A \to \perp,$$

Nelson's approach takes negation as a primitive operator endowed with its own proof-theoretic and semantics content. This allows for a distinction between the absence of a proof of A and the presence of a constructive refutation of A.

The SR System: Introduction

Nelson's paraconsistent system N4, provide a setting in which both A and its negation $\sim A$ may be supported without triviality, and offers a constructive account of falsity.

This feature makes Nelson's negation especially useful for the study of logics that reject explosion and enforce a more refined connection between premises and conclusions, such as relevance logics.

As such, constructive negation serves as a natural bridge between constructive and paraconsistent traditions, and provides a robust foundation for analyzing negation in contexts of incomplete or inconsistent information.

The SR System: Syntax

In order to obtain the system SR extending the system SR^+ with a constructive negation based on Nelson (1949), we add the following rules:

Definition 16: Derivation Rules for Negation

$$\frac{A_\alpha \quad \sim B_\alpha}{\sim (A \to B)_\alpha}\ [\sim\!\to I] \qquad \frac{\sim (A \to B)_\alpha}{A_\alpha}\ [\sim\!\to E_1] \qquad \frac{\sim (A \to B)_\alpha}{\sim B_\alpha}\ [\sim\!\to E_2]$$

$$\frac{\sim A_\alpha}{\sim (A \wedge B)_\alpha}\ [\sim \wedge I_1] \qquad \frac{\sim B_\alpha}{\sim (A \wedge B)_\alpha}\ [\sim \wedge I_2] \qquad \frac{\sim (A \wedge B)_\alpha \quad \begin{array}{c} \sim A_\alpha \\ \vdots \\ C_{\alpha\cup\beta} \end{array} \quad \begin{array}{c} \sim B_\alpha \\ \vdots \\ C_{\alpha\cup\beta} \end{array}}{C_{\alpha\cup\beta}}\ [\sim \wedge E]$$

$$\frac{\sim A_\alpha \quad \sim B_\alpha}{\sim (A \vee B)_\alpha}\ [\sim \vee I] \qquad \frac{\sim (A \vee B)_\alpha}{\sim A_\alpha}\ [\sim \vee E_1] \qquad \frac{\sim (A \vee B)_\alpha}{\sim B_\alpha}\ [\sim \vee E_2]$$

$$\frac{A_\alpha}{\sim\sim A_\alpha}\ [\sim\sim I] \qquad \frac{\sim\sim A_\alpha}{A_\alpha}\ [\sim\sim E]$$

The SR System: Syntax

We can introduce two derivability relations for SR.

Definition 17: Strict Derivability in SR

A formula A of SR is **strictly derivable** in SR from a set of formulas $\Gamma = \{B_1, \ldots, B_n\}$ of SR, denoted by $\Gamma \vdash_{\mathsf{SR}} A$, if for some $\{k_1, \ldots, k_n\} \in \mathcal{P}(\mathbb{N})$, it follows that $B_{1_{\{k_1\}}}, \ldots, B_{n_{\{k_n\}}} \vdash_{\mathsf{SR}} A_\alpha$, where $\alpha = \{k_1, \ldots, k_n\}$.

Definition 18: Non-Strict Derivability in SR

A formula $A \in \mathcal{F}_{\mathsf{SR}}$ is **non-strictly derivable** in SR from a set of formulas $\Gamma \subseteq \mathcal{F}_{\mathsf{SR}}$, denoted by $\Gamma \Vdash_{\mathsf{SR}} A$, if there exists a finite subset $\{B_1, \ldots, B_n\} \subseteq \Gamma$ such that $B_1, \ldots, B_n \vdash_{\mathsf{SR}} A$.

The SR System: Semantics

Definition 19: Semilattice Model for SR

A **semilattice model for** SR is a structure $\langle S, \sqcup, 0, v^+, v^- \rangle$, where $\langle S, \sqcup, 0 \rangle$ is a semilattice frame, and $v^+ : V \to \mathcal{P}(S)$ and $v^- : V \to \mathcal{P}(S)$ are the verification and falsification valuation function respectively.

The SR System: Semantics

Definition 20: Verification and Falsification Relations

Given a semilattice model $\mathcal{M} = \langle S, \sqcup, 0, v^+, v^- \rangle$ for SR, the **verification and falsification relations**, denoted by $\models^+_{\mathcal{M}}$ and $\models^-_{\mathcal{M}}$, between elements of S and formulas of SR, are simultaneously defined by:

$$x \models^*_{\mathcal{M}} p \text{ iff } x \in v^*(p), \text{ for all } p \in V,$$

$$x \models^+_{\mathcal{M}} \sim A \text{ iff } x \models^-_{\mathcal{M}} A,$$

$$x \models^+_{\mathcal{M}} A \to B \text{ iff for all } y \in S, \text{ whenever } y \models^+_{\mathcal{M}} A \text{ then } x \sqcup y \models^+_{\mathcal{M}} B,$$

$$x \models^+_{\mathcal{M}} A \wedge B \text{ iff } x \models^+_{\mathcal{M}} A \text{ and } x \models^+_{\mathcal{M}} B,$$

$$x \models^+_{\mathcal{M}} A \vee B \text{ iff } x \models^+_{\mathcal{M}} A \text{ or } x \models^+_{\mathcal{M}} B,$$

$$x \models^-_{\mathcal{M}} \sim A \text{ iff } x \models^+_{\mathcal{M}} A,$$

$$x \models^-_{\mathcal{M}} A \to B \text{ iff } x \models^+_{\mathcal{M}} A \text{ and } x \models^-_{\mathcal{M}} B,$$

$$x \models^-_{\mathcal{M}} A \wedge B \text{ iff } x \models^-_{\mathcal{M}} A \text{ or } x \models^-_{\mathcal{M}} B,$$

$$x \models^-_{\mathcal{M}} A \vee B \text{ iff } x \models^-_{\mathcal{M}} A \text{ and } x \models^-_{\mathcal{M}} B.$$

The SR System: Semantics

Definition 21

A formula $A \in \mathcal{F}_{\mathsf{SR}}$ is **verified** (respectively, **falsified**) in a semilattice model $\mathcal{M} = \langle S, \sqcup, 0, v^{+}, v^{-} \rangle$ for SR if $0 \models^{+}_{\mathcal{M}} A$ (respectively, $0 \models^{-}_{\mathcal{M}} A$).

Definition 22

A formula $A \in \mathcal{F}_{\mathsf{SR}}$ is **valid** in the semilattice semantics for SR, denoted by $\models_{\mathsf{SR}} A$, if A is verified in all semilattice models for SR.

The SR System: Adequacy Theorems

To establish weak and strong adequacy theorems for the semilattice semantics of SR, we follow the method of Kamide and Wansing (2012):

1. Introduce a translation function from formulas of SR into formulas of SR^+.
2. Construct both syntactic and semantic embeddings of SR into SR^+.
3. By appealing to the Weak Adequacy Theorem for the Semilattice Semantics of SR^+ (Theorem 3), obtain soundness and completeness results.

Definition 23

Define the set $V' = \{p' \mid p \in V\}$ of propositional variables.

- The **language** $\mathcal{L}_{\mathrm{SR}}$ **of** SR is defined using V and the connectives $\rightarrow$, $\wedge$, $\vee$, and $\sim$.
- The **language** $\mathcal{L}_{\mathrm{SR}^+}$ **of** SR^+ is obtained from $\mathcal{L}_{\mathrm{SR}}$ by adding V' and deleting $\sim$.

The SR System: Adequacy Theorems

Definition 24: Translation Function from $\mathcal{L}_{\mathsf{SR}}$ to $\mathcal{L}_{\mathsf{SR}^+}$

The **translation** T *from* $\mathcal{L}_{\mathsf{SR}}$ *to* $\mathcal{L}_{\mathsf{SR}^+}$ is a mapping inductively defined by:

$$T(p) := p \text{ and } T(\sim p) := p', \text{ for all } p \in V,$$
$$T(A \circ B) := T(A) \circ T(B), \text{ where } \circ \in \{\rightarrow, \wedge, \vee\},$$
$$T(\sim\sim A) := T(A),$$
$$T(\sim (A \rightarrow B)) := T(A) \wedge T(\sim B),$$
$$T(\sim (A \wedge B)) := T(\sim A) \vee T(\sim B),$$
$$T(\sim (A \vee B)) := T(\sim A) \wedge T(\sim B).$$

Remark 9

An expression $T(\Gamma)$ denotes the result of replacing every occurrence of a formula A in Γ by an occurrence of $T(A)$.

The SR System: Adequacy Theorems

Theorem 4: Syntactical Embedding

Let $\Gamma = \{B_1, \ldots, B_n\}$ be a set of formulas in $\mathcal{L}_{\mathsf{SR}}$, A a formula in $\mathcal{L}_{\mathsf{SR}}$, and T the mapping defined in Definition 24. Then $\Gamma \vdash_{\mathsf{SR}} A$ if and only if $T(\Gamma) \vdash_{\mathsf{SR}^+} T(A)$.

Proof. By induction on the high h of the derivation tree.

Theorem 5: Semantical Embedding

For any formula $A \in \mathcal{L}_{\mathsf{SR}}$, A is valid in SR if and only if $T(A)$ is valid in SR^+.

Proof. By structural induction on $A \in \mathcal{L}_{\mathsf{SR}}$ and $T(A) \in \mathcal{L}_{\mathsf{SR}^+}$.

The SR System: Adequacy Theorems

Theorem 6: Weak Adequacy Theorem for the Semilattice Semantics for SR

Let A be an formula in SR. Then $\vdash_{\mathsf{SR}} A$ if and only if $\models_{\mathsf{SR}} A$.

Proof. It is a direct consequence of Theorems 4 and 5.

Theorem 7: Deduction Theorem for SR

Let $\Gamma = \{B_1, \ldots, B_n\}$ a set of formulas in SR and A a formula in SR. Then $B_{1_{\{k_1\}}}, \ldots, B_{n_{\{k_n\}}} \vdash_{\mathsf{SR}} A_\alpha$ if and only if $B_{1_{\{k_1\}}}, \ldots, B_{n-1_{\{k_{n-1}\}}} \vdash_{\mathsf{SR}} (B_n \to A)_{\alpha \setminus \{k_n\}}$, where $\alpha = \{k_1, \ldots, k_n\}$.

Proof. Immediately by induction on the length of Γ.

The SR System: Adequacy Theorems

Definition 25: Strict Semantical Consequence in a Semilattice Model $\mathcal{M}$ for SR

Let $\Gamma = \{B_1, \ldots, B_n\}$ be a set of formulas in SR and $A \in \mathcal{L}_{\mathsf{SR}}$. We say that A is a **strict semantical consequence of Γ in a semilattice model** $\mathcal{M} = \langle S, \sqcup, 0, v^+, v^- \rangle$ for SR, denoted by $\Gamma \models_{\mathcal{M}} A$, if for all $x_1, \ldots, x_n \in S$, whenever $x_1 \models^+_{\mathcal{M}} B_1, \ldots, x_n \models^+_{\mathcal{M}} B_n$, it follows that $x_1 \sqcup \cdots \sqcup x_n \models^+_{\mathcal{M}} A$.

Definition 26: Strict Semantical Consequence in the Semilattice Semantics for SR

Let $\Gamma = \{B_1, \ldots, B_n\}$ be a set of formulas in SR and $A \in \mathcal{L}_{\mathsf{SR}}$. We say that A is a **strict semantical consequence of Γ in the semilattice semantics for** SR, denoted by $\Gamma \models_{\mathsf{SR}} A$, if $\Gamma \models_{\mathcal{M}} A$ holds in every semilattice model $\mathcal{M} = \langle S, \sqcup, 0, v^+, v^- \rangle$ for SR.

The SR System: Adequacy Theorems

Definition 27: Non-Strict Semantical Consequence in a Semilattice Model $\mathcal{M}$ for SR

Let Γ be a set of formulas in SR and $A \in \mathcal{L}_{\mathsf{SR}}$. We say that A is a **non-strict semantical consequence of Γ in a semilattice model** $\mathcal{M} = \langle S, \sqcup, 0, v^{+}, v^{-} \rangle$ for SR, denoted by $\Gamma \Vert\!\models_{\mathcal{M}} A$, if there exists a finite subset $\{B_1, \ldots, B_n\} \subseteq \Gamma$ such that $B_1, \ldots, B_n \models_{\mathcal{M}} A$.

Definition 28: Non-Strict Semantical Consequence in the Semilattice Semantics for SR

Let Γ be a set of formulas in SR and $A \in \mathcal{L}_{\mathsf{SR}}$. We say that A is a **non-strict semantical consequence of Γ in the semilattice semantics for** SR, denoted by $\Gamma \Vert\!\models_{\mathsf{SR}} A$, if there exists a finite subset $\{B_1, \ldots, B_n\} \subseteq \Gamma$ such that $B_1, \ldots, B_n \models_{\mathcal{M}} A$ holds in every semilattice model $\mathcal{M} = \langle S, \sqcup, 0, v^{+}, v^{-} \rangle$ for SR.

The SR System: Adequacy Theorems

Theorem 8

Let $\Gamma = \{B_1, \ldots, B_n\}$ be a set of formulas in SR and A be a formula in SR. Then $B_1, \ldots, B_n \models_{\mathsf{SR}} A$ if and only if $B_1, \ldots, B_{n-1} \models_{\mathsf{SR}} B_n \to A$.

Proof. By distinguishing cases based on the size of n of Γ.

Theorem 9: Strong Adequacy Theorem for the Strict Semilattice Semantics for SR

Let $\Gamma = \{B_1, \ldots, B_n\}$ be a set of formulas in SR and A be a formula in SR. Then $\Gamma \vdash_{\mathsf{SR}} A$ if and only if $\Gamma \models_{\mathsf{SR}} A$.

Proof. By induction on n.

The SR System: Adequacy Theorems

Remark 10

The strong version of the Adequacy Theorem for the Semilattice Semantics of SR holds only when the set of formulas Γ is finite, since it is based on the strict derivability relation. This derivability relation is not monotonic, as shown by the following theorems.

The SR System: Adequacy Theorems

Theorem 10

The strict semantic relation in SR, $\models_{\mathsf{SR}}$, is not monotonic.

Proof. Consider the following semilattice model $\mathcal{M} = \langle S, \sqcup, 0, v^+, v^- \rangle$, where

- $S = 0, t, u, w$,
- $\sqcup : S \times S \to S$ is defined by:
  - $0 \sqcup x = x$ for all $x \in S$,
  - $x \sqcup x = x$ for all $x \in S$,
  - $t \sqcup u = w$,
- $v^+(p) = \{t\}$ and $v^+(q) = \{u\}$,
- $v^-(p) = v^-(q) = \emptyset$.

Then $t \models^+_{\mathcal{M}} p$ and $u \models^+_{\mathcal{M}} q$, but $t \sqcup u = w \not\models^+_{\mathcal{M}} p$. Hence, $p, q \not\models_{\mathsf{SR}} p$, and it follows that $\models_{\mathsf{SR}}$ is not monotonic.

The SR System: Adequacy Theorems

Theorem 11

The strict derivability relation in SR, $\vdash_{SR}$, is not monotonic.

Proof. Follows immediately by the Strong Adequacy Theorem for the Strict Semilattice Semantics and Theorem 10.

Remark 11

Despite the non-monotonicity of $\vdash_{SR}$, we can extend the strong adequacy result to arbitrary (possibly infinite) sets of formulas by means of the non-strict derivability relation in SR, $\Vdash_{SR}$, as shown in the following theorem.

The SR System: Adequacy Theorems

Theorem 12: Strong Adequacy Theorem for the Non-Strict Semilattice Semantics for SR

Let Γ be a set of formulas (possibly infinite) in SR and A be a formula in SR. Then, $\Gamma \Vdash_{\mathsf{SR}} A$ if and only if $\Gamma \| \!\models_{\mathsf{SR}} A$.

Proof. By definition of $\Vdash_{\mathsf{SR}}$, $\Gamma \Vdash_{\mathsf{SR}} A$ iff there exists a finite subset $\{B_1, \ldots, B_n\} \subseteq \Gamma$ such that $B_1, \ldots, B_n \vdash_{\mathsf{SR}} A$. By the Strong Adequacy Theorem for the Strict Semilattice Semantics for SR, it holds iff $B_1, \ldots, B_n \models_{\mathsf{SR}} A$. Finally, by definition of $\| \!\models_{\mathsf{SR}}$, we have that it holds iff $\Gamma \| \!\models_{\mathsf{SR}} A$.

The SR System: Conservativity and Undecidability

Another open problem proposed by Anderson in 1963 was the decision problem.

- Kripke (1959) established the decidability of $R_{\rightarrow}$.
- Meyer extended Kripke's method to the system LR (i.e., R without the distribution axiom) (Meyer, 1966).
- The problem for R remained open until 1984, when Urquhart (1984) showed that these systems are undecidable.
- Recently, Knudstorp (2024) proved that SR^{+} is undecidable.

By exploiting the Adequacy Theorem for the Semilattice Semantics of SR, together with the embedding of SR into SR^{+} via the translation function, we can conclude that SR is also undecidable.

The SR System: Conservativity and Undecidability

Theorem 13: Conservativity

For every formula A in SR^+, $\models_{\mathsf{SR}^+} A$ if and only if $\models_{\mathsf{SR}} A$.

Proof. Follows immediately by the Semantical Embedding and the fact that for every formula A in SR^+ based on the set of propositional variables V, we have that $A = T(A)$.

Theorem 14: Undecidability of SR

SR is undecidable.

Proof. Follows from the Conservativity Theorem and the undecidability of SR^+.

The SR System: Paradoxes and Other Properties

Relevance logic systems were developed to eliminate the paradoxes of material implication. As Urquhart (1972) observes, these paradoxes can be understood, as falling into two distinct classes.

- The admissibility of the first class depends on the acceptability of introducing irrelevant hypotheses.
- The admissibility of the second class depends on whether inconsistent or incomplete pieces of information are admitted or excluded.

In the case of SR:

- Paradoxes of the first class are excluded from the system thanks to the subscripting device.
- Paradoxes from the second class are also avoided due to the introduction of Nelson's negation.

Other properties of SR include the validation of double negation and De Morgan laws, and the exclusion of reductio, contraposition and the Ackermann's rule γ.

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