On discussive logics defined on a connexive modal logic[1]

Ricardo Arturo Nicolás-Francisco
(joint work with Marek Nasieniewski)

Nicolaus Copernicus University and UNAM

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This is a work still in progress. The basic idea of the work is that a connexive discussive logic can be formulated by referring to a connexive modal logic, similarly as Jaśkowski's discussive logic was formulated by referring to a modal logic based on classical logic.

Plan

Connexive logic

Connexive logic is a logic associated with the following principles:

$$
\begin{array}{ll}
N(A > NA) & \text{(Aristotle's Thesis)} \\
N(NA > A) & \text{(Variant of Aristotle's Thesis)} \\
(A > B) > N(A > NB) & \text{(Boethius' Thesis)} \\
(A > NB) > N(A > B) & \text{(Variant of Boethius' Thesis)}
\end{array}
$$

where N is considered to be a negation and $>$ is considered to be a conditional.

Furthermore, $(A > B) =_{def} N(A \circ NB)$, where $\circ$ denotes compatibility.

A propositional calculus for inconsistent deductive systems

In [Jaśkowski, 1948/1999] Jaśkowski investigated the possibility to devise a logic with the following properties:

1. that when applied to inconsistent systems do not entail its triviality (or 'overcompleteness', as Jaśkowski called it),
2. that would be rich enough to allow for practical inferences,
3. that would have intuitive justification.

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> *"A deductive system S is called inconsistent if its theses include two such which contradict one another, that is, such that one is the negation of the other, e.g., A and N A" (notation adjusted)[Jaśkowski, 1948/1999, p.38]*

The concept of inconsistency

"The concept of an inconsistency is by no means a sharp one. Besides logical contradictions there are refutations, collisions of thesis and antithesis, incoherences, antinomies, contrasts, oppositions, self-defeating assertions ... I just take the word "inconsistency" to be a generic term for all kinds of incompatibility and polarity"[Urchs(1999), p.137]

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We can define the notion of incompatibility (denoted as $\bullet$) in terms of negation and implication as follows:

$$A \bullet B =_{def} N(A \circ B) =_{def} (A > NB).$$

Assuming 1) $(A > B) > (A \circ B)$, and 2) $>$ is contrapositive, we can prove $(A > NB) > N(A > B)$.

Connexive logic C

Let $\mathcal{L}$ be a propositional language built from a denumerable set of propositional variables *Var* and the set of connectives $\{\wedge, \vee, \rightarrow, \sim\}$. Let For be the set of formulas of $\mathcal{L}$.

The logic C was presented by Wansing in [Wansing, 2005] with the following axiom schemes and derivation rules, where $\leftrightarrow$ is defined as a conjunction of two implications:

- all axioms schemes of intuitionistic positive logic.
- $\sim\sim A \leftrightarrow A$
- $\sim(A \vee B) \leftrightarrow (\sim A \wedge \sim B)$
- $\sim(A \wedge B) \leftrightarrow (\sim A \vee \sim B)$
- $\sim(A \rightarrow B) \leftrightarrow (A \rightarrow \sim B)$
- modus ponens,

where the notion of derivability and consequence relation $\vdash_C$ are defined standardly.

Connexive logic CK

Let $\mathcal{L}_m$ be a propositional language built from a denumerable set of propositional variables *Var* and the set of connectives $\{\wedge, \vee, \rightarrow, \sim, \Box, \Diamond\}$. Let For_m be the set of formulas of $\mathcal{L}_m$.

The modal connexive logic CK is built in For_m and extends the logic C with the following axioms schemes and rules (where $\leftrightarrow$ is defined as a conjunction of two implications):

$$\Box A \wedge \Box B \rightarrow \Box(A \wedge B) \tag{K2}$$

$$\Box(A \rightarrow A) \tag{L$\top$}$$

$$\Diamond(A \vee B) \rightarrow \Diamond A \vee \Diamond B \tag{M6}$$

$$\Diamond(A \rightarrow B) \rightarrow (\Box A \rightarrow \Diamond B) \tag{K7}$$

$$(\Diamond A \rightarrow \Box B) \rightarrow \Box(A \rightarrow B) \tag{KT$\Box$}$$

$$\sim \Box A \leftrightarrow \Diamond \sim A \tag{Df$\Diamond$}$$

$$\sim \Diamond A \leftrightarrow \Box \sim A \tag{Df$\Box$}$$

Connexive logic CK

$$A \to B / \Box A \to \Box B \tag{$\mathrm{RM}_\Box$}$$

$$A \to B / \Diamond A \to \Diamond B \tag{$\mathrm{RM}_\Diamond$}$$

CK is a connexive analogue of the normal modal logic K.

Connexive logic CK^+

We will consider an extension of the modal connexive logic CK with an axiom (K9) (see below). We denote the resulting extension CK^+:

$$\Box A \wedge \Box B \to \Box(A \wedge B) \tag{K2}$$

$$\Box(A \to A) \tag{L$\top$}$$

$$\Diamond(A \vee B) \to \Diamond A \vee \Diamond B \tag{M6}$$

$$\Diamond(A \to B) \to (\Box A \to \Diamond B) \tag{K7}$$

$$(\Diamond A \to \Box B) \to \Box(A \to B) \tag{KT$\Box$}$$

$$\sim \Box A \leftrightarrow \Diamond \sim A \tag{Df$\Diamond$}$$

$$\sim \Diamond A \leftrightarrow \Box \sim A \tag{Df$\Box$}$$

$$\Box(A \vee B) \to \Box A \vee \Diamond B \tag{K9}$$

A new logic CD

We extend the modal logic CK^+ with the following axioms to obtain the connexive analogue of the normal modal logic D:

$$\Box A \to \Diamond A \tag{D}$$

$$\Diamond(A \to A) \tag{D1}$$

We call the resulting logic CD.

CK$^+$-frames

A *CK$^+$-frame* is a triple $\mathcal{F} = \langle W, R, \leq \rangle$, where:

- W is a non-empty set of information states,
- R is an arbitrary binary relation on W,
- $\leq$ is a binary relation on W such that:

$$\leq \text{ is reflexive and transitive} \tag{RT}$$

$$\leq^{-1} \circ R \subseteq R \circ \leq^{-1} \tag{C1}$$

$$R \circ \leq \subseteq \leq \circ R \tag{C2}$$

$$\leq \circ R \subseteq R \circ \leq \tag{C3}$$

where $\leq^{-1}$ is a converse of $\leq$, while $\circ$—the standard composition of binary relations.

CK^+-models

Let $\langle W, \leq \rangle^+$ be the set of all $X \subseteq W$ such that if $u \in X$ and $u \leq w$, then $w \in X$.

A *CK^+-model* is a structure $\mathcal{M} = \langle W, R, \leq, v^+, v^- \rangle$ where $\langle W, R, \leq \rangle$ is a CK^+-frame and v^+ and v^- are valuation functions from *Var* into $\langle W, \leq \rangle^+$.

The function v^+ assigns to each propositional variable p, the set of states in W that support the truth of p, whereas v^- assigns to p, the states that support the falsity of p.

CD-frames and CD-models

A *CD-frame* is a CK^+-frame where R is serial, that is:

for every $u \in W$, there is a $w \in W$ such that uRw (seriality)

A *CD-model* is a structure $\mathcal{M} = \langle W, R, \leq, v^+, v^- \rangle$ where:

- $\langle W, R, \leq \rangle$ is a CD-frame
- v^+ and v^- are valuation functions from Var into $\langle W, \leq \rangle^+$ as in a CK^+-model.

Truth and falsity conditions

The support of truth and falsity of a formula A in a state w of a model $\mathcal{M}$, denoted respectively by $\mathcal{M}, w \models^{+} A$ and $\mathcal{M}, w \models^{-} A$, are defined as follows, where $a \in Var$:

$$\mathcal{M}, w \models^{+} p \text{ iff } w \in v^{+}(p) \qquad (\mathrm{Var}^{+})$$

$$\mathcal{M}, w \models^{-} p \text{ iff } w \in v^{-}(p) \qquad (\mathrm{Var}^{-})$$

$$\mathcal{M}, w \models^{+} (A \wedge B) \text{ iff } \mathcal{M}, w \models^{+} A \text{ and } \mathcal{M}, w \models^{+} B \qquad (\wedge^{+})$$

$$\mathcal{M}, w \models^{-} (A \wedge B) \text{ iff } \mathcal{M}, w \models^{-} A \text{ or } \mathcal{M}, w \models^{-} B \qquad (\wedge^{-})$$

$$\mathcal{M}, w \models^{+} (A \vee B) \text{ iff } \mathcal{M}, w \models^{+} A \text{ or } \mathcal{M}, w \models^{+} B \qquad (\vee^{+})$$

$$\mathcal{M}, w \models^{-} (A \vee B) \text{ iff } \mathcal{M}, w \models^{-} A \text{ and } \mathcal{M}, w \models^{-} B \qquad (\vee^{-})$$

Truth and falsity conditions

$$\mathcal{M}, w \models^{+} (A \to B) \text{ iff } \forall w' \geq w(\mathcal{M}, w' \models^{+} A \Rightarrow \mathcal{M}, w' \models^{+} B) \qquad (\to^{+})$$

$$\mathcal{M}, w \models^{-} (A \to B) \text{ iff } \forall w' \geq w(\mathcal{M}, w' \models^{+} A \Rightarrow \mathcal{M}, w' \models^{-} B) \qquad (\to^{-})$$

$$\mathcal{M}, w \models^{+} {\sim} A \text{ iff } \mathcal{M}, w \models^{-} A \qquad ({\sim}^{+})$$

$$\mathcal{M}, w \models^{-} {\sim} A \text{ iff } \mathcal{M}, w \models^{+} A \qquad ({\sim}^{-})$$

$$\mathcal{M}, w \models^{+} \Box A \text{ iff } \forall u \geq w \forall u'(uRu' \text{ implies } \mathcal{M}, u' \models^{+} A) \qquad (\Box^{+})$$

$$\mathcal{M}, w \models^{-} \Box A \text{ iff } \exists u(wRu \text{ and } \mathcal{M}, u \models^{-} A) \qquad (\Box^{-})$$

$$\mathcal{M}, w \models^{+} \Diamond A \text{ iff } \exists u(wRu \text{ and } \mathcal{M}, u \models^{+} A) \qquad (\Diamond^{+})$$

$$\mathcal{M}, w \models^{-} \Diamond A \text{ iff } \forall u \geq w \forall u'(uRu' \text{ implies } \mathcal{M}, u' \models^{-} A) \qquad (\Diamond^{-})$$

Validity

The notions of validity in a CD-model — $\mathcal{M} \models A$, and validity in a CD-frame — $\mathcal{F} \models A$, are defined in the usual way.

A formula A is CD-valid [written: $\models_{\mathsf{CD}} A$] iff it is valid in every CD-frame.

Hereditary property

Lemma (Hereditariness for CK)

For any CK-model $\mathcal{M} = \langle W, \leq, R, v \rangle$, formula A, and any $s, t \in \mathcal{W}$, if $s \leq t$, then $\mathcal{M}, s \models^{+} A$ implies $\mathcal{M}, t \models^{+} A$ and $\mathcal{M}, s \models^{-} A$ implies $\mathcal{M}, t \models^{-} A$.

Lemma (Hereditariness for CD)

For any CD-model $\mathcal{M} = \langle W, \leq, R, v \rangle$, formula A, and any $s, t \in \mathcal{W}$, if $s \leq t$, then $\mathcal{M}, s \models^{+} A$ implies $\mathcal{M}, t \models^{+} A$ and $\mathcal{M}, s \models^{-} A$ implies $\mathcal{M}, t \models^{-} A$.

Theorem ([Wansing, 2005, Odintsov and Wansing, 2004].)

For any A, $A \in CK$ iff $\models_{CK} A$.

Soundness for CD

Theorem

For any A, if $A \in CD$, then $\models_{CD} A$.

CD-prime sets

A set of formulas is called *CD-prime* if:

1. it contains all CD-theorems,
2. it is closed under MP
3. it possesses the disjunction property in the form:

$$\text{if } \Gamma \vdash A \vee B, \text{ then } \Gamma \vdash A \text{ or } \Gamma \vdash B.$$

For the set of formulas Γ and Δ, the notation $\Gamma \vdash \Delta$ means that

$$\vdash A_1 \wedge \ldots \wedge A_n \rightarrow B_1 \vee \ldots \vee B_m$$

Fact

1. *If* $\Gamma \nvDash \Delta$ *and* $\Sigma \subseteq \Gamma$, *then* $\Sigma \nvDash \Delta$
2. $\Gamma \nvDash \Delta$ *iff for every finite subset* Σ *of* Γ, $\Sigma \nvDash \Delta$.

Fact (CD-prime sets — theories)

Let Γ *be a CD-prime set. If* $\Gamma \vdash A$ *then* $A \in \Gamma$.

Lemma (Lindenbaum-Asser lemma)

If $\Gamma \nvDash \Delta$, *then there exists a CD-prime set* Γ' *such that* $\Gamma \subseteq \Gamma'$ *and* $\Gamma' \nvDash \Delta$. *Moreover, for such* Γ', *for every* A, *if* $\Gamma' \vdash A$ *then* $A \in \Gamma'$ *(or equivalently* $\Gamma' \cup \{A\} \nvDash \Delta$, *then* $A \in \Gamma'$*).*

The proof of the following Lemma uses an observation of Odintsov and Wansing from [Odintsov and Wansing, 2004, p.275].

Lemma

Let Δ be a CD-prime set. $\Delta^{\Diamond} \subseteq \Gamma$ is equivalent to $\Delta \not\models \{A | \Diamond A \notin \Gamma\}$.

One can also prove the following lemma:

Lemma

Let Δ be a CD-prime set. $\Delta' \not\models \{\Box A : A \notin \Delta\}$ is equivalent to $\Delta'_{\Box} \subseteq \Delta$.

Lemma

The canonical CD-model is a CD-model.

Let $\mathrm{Cl}(X) = \{A : X \vdash A\}$ and W^c be the family of all CD-prime sets.
We have:

Lemma ([Odintsov and Wansing, 2004, Lemma 6, p. 276])

The following facts are true for any $\Gamma \in W^c$ and formula A.

1. *If $\Gamma_{\Box} \vdash A$, then $A \in \Gamma_{\Box}$*
2. $\Gamma_{\Box} = (\mathrm{Cl}(\Gamma \cup \Diamond(B \to B)))_{\Box}$

Lemma (Canonical Lemma)

For a formula A and any $\Gamma \in W^c$,

$$\mathcal{M}^c, \Gamma \models^+ A \textit{ iff } A \in \Gamma; \ \mathcal{M}^c, \Gamma \models^- A \textit{ iff } {\sim} A \in \Gamma.$$

Theorem (Completeness)

For any formula A:

$$\models_{CD} A \text{ implies } \vdash_{CD} A.$$

For a given modal logic S, let $\Diamond$-$S = \{A \in Form : \Diamond A \in S\}$.

It is a known fact (see [Perzanowski, 1975, p. 68]) that for the standard normal modal logic D, $\Diamond$-D$=$ D.

We show that we have a similar result for the logic CD, that is $\Diamond$-CD$=$CD.

Theorem

$CD = \Diamond$-CD

A prospect of a connexive discussive logic

We will consider a variation of discussive logic based on CD, so we consider a second propositional language $\mathcal{L}_d$ built from the set of propositional variables Var and the set of connectives $\{\sim, \vee, \rightarrow_d, \wedge_d\}$. We denote the resulting set of discussive formulas by $\mathrm{For}_{\mathrm{d}}$.

Let us consider a function: i_1 from $\mathrm{For}_{\mathrm{d}}$ to For_m satisfying the following conditions:

1. $i_1(a) = a$, for any propositional variable a
2. for any $A, B \in \mathrm{For}_{\mathrm{d}}$
   - $i_1(\sim A) = \sim i_1(A)$
   - $i_1(A \vee B) = i_1(A) \vee i_1(B)$
   - $i_1(A \wedge_d B) = i_1(A) \wedge \Diamond i_1(B)$
   - $i_1(A \rightarrow_d B) = \Diamond i_1(A) \rightarrow i_1(B)$.

Now we can define a discussive logic based on CD.

The logic $\mathbf{Disc}_{\mathsf{CD}}$ is the following subset of $\mathrm{For}_{\mathrm{d}}$:

$$\mathbf{Disc}_{\mathsf{CD}} := \{\, A \in \mathrm{For}_{\mathrm{d}} : \ulcorner \Diamond i_1(A) \urcorner \in \mathsf{CD} \,\}.$$

Taking into account Theorem 14, we can reformulate the above definition be stipulating:

$$\mathbf{Disc}_{\mathsf{CD}} := \{\, A \in \mathrm{For}_{\mathrm{d}} : \ulcorner i_1(A) \urcorner \in \mathsf{CD} \,\}.$$

It is the weakest logic in some family.

Some prospects of a discussive logic based on a connexive logic

Examples of non-valid formulas:

- $p \rightarrow_d p$
- $p \rightarrow_d (q \rightarrow_d (p \wedge q))$
- $((p \wedge q) \rightarrow_d r) \rightarrow_d (p \rightarrow_d (q \rightarrow_d r))$
- $p \rightarrow_d (\sim p \rightarrow_d q)$

Examples of valid arguments:

- $p, \sim p \models p \rightarrow_d \sim p$
- $p, \sim p \models \sim p \rightarrow_d p$
- $p \models q \rightarrow_d p$
- $(p \wedge_d q) \models (q \rightarrow_d p)$

Summary

- We expanded the modal logic CK to obtain the modal logic CK^{+}, and then the logic CD.
- Using a similar translation as the one used by Jaśkowski, we considered a possible formulation of discussive logic based on the connexive modal logic CD.
- Given the similarity of introducing this discussive logic with the way Jaśkowski's (indirectly) introduced $\mathsf{D_2}$, we will also propose a syntactic formulation —more direct— of the proposed discussive logic.

Thank you very much!

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The search for an implication in discussive logic

The introduction of discussive logic creates a problem of finding an implication:

> *. . . in the search for a "logic of discussion" the prime task is to choose such a function which . . . would play the role analogous to that which in ordinary systems is played by implication. The problem, if formulated in this way, has a number of solutions . . . . Each solution would yield a different system of discussive logic. [Jaśkowski, 1948/1999, p. 44]*

The search for an implication in discussive logic

- Jaśkowski presented one solution by characterizing the implication in a specific way that can validate modus ponens.
- However, Jaśkowski himself observed that his solution resulted in one discussive logic, but he did not explore all implications.
- Since connexive logics has to do mainly with implication, a possible solution, not explore yet, is suggested from there.