A characterization of the partial orders associated to right-normal bands

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Introduction

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Key fact

This preorder is a partial order if and only if the algebra satisfies

$$x \cdot y \cdot x = y \cdot x.$$

Bands satisfying this identity are called *right-regular bands*.

Convention

Let $(A, \cdot)$ be a right-regular band. We write

$$x \leq y \quad \text{iff} \quad x \cdot y = x.$$

A poset associated with such an algebra will be called an *associative poset*.

Lemma

Let A be a right-regular band and $\leq$ its associated order.
Then for every $a, b \in A$,

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Corollary

If B is a downward closed subset of A, then B is a subuniverse.

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$$f : A \to B$$

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If $f : A \to B$ is an isomorphism of right-regular bands, then it is also an order isomorphism.

Quotient Semilattice

Lemma/Definition

Let $(A, \cdot)$ be a right-regular band. The binary relation $\theta := \{(x, y) \in A^2 : (x \cdot y = y) \;\&\; (y \cdot x = x)\}$ is a congruence on A, and A/θ is a semilattice. We call θ the semilattice congruence of A and A/θ the quotient semilattice of A.

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To see that it is a congruence, let $(x_1, x_2), (y_1, y_2) \in \theta$. Then
$(x_1 \cdot y_1) \cdot (x_2 \cdot y_2) = (x_1 \cdot x_2 \cdot y_1 \cdot y_2) \cdot (x_2 \cdot y_2) = (x_2 \cdot y_2) \cdot (x_2 \cdot y_2) = x_2 \cdot y_2$.

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Similarly, one proves that $(x_2 \cdot y_2) \cdot (x_1 \cdot y_1) = x_1 \cdot y_1$, and therefore $((x_1 \cdot y_1), (x_2 \cdot y_2)) \in \theta$, so θ is a congruence.

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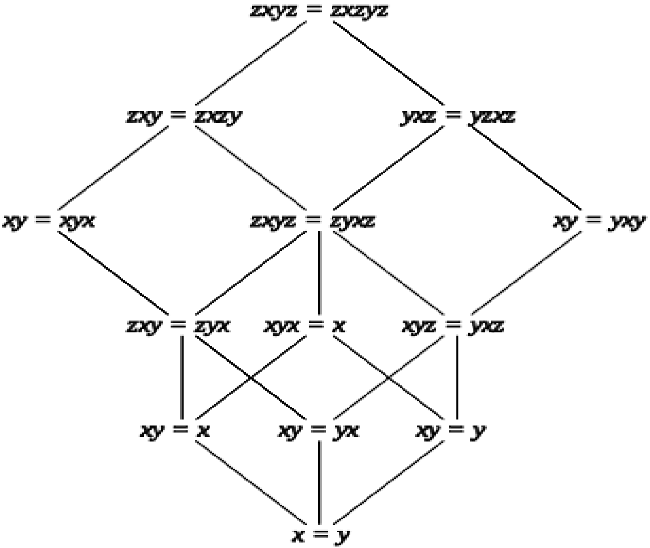
To see that A/θ is a semilattice, note that for all $x, y \in A$ we have $(x \cdot y) \cdot (y \cdot x) = y \cdot x$ and $(y \cdot x) \cdot (x \cdot y) = x \cdot y$. Hence

$$x/\theta \cdot y/\theta = (x \cdot y)/\theta = (y \cdot x)/\theta = y/\theta \cdot x/\theta.$$

Subvarieties of bands



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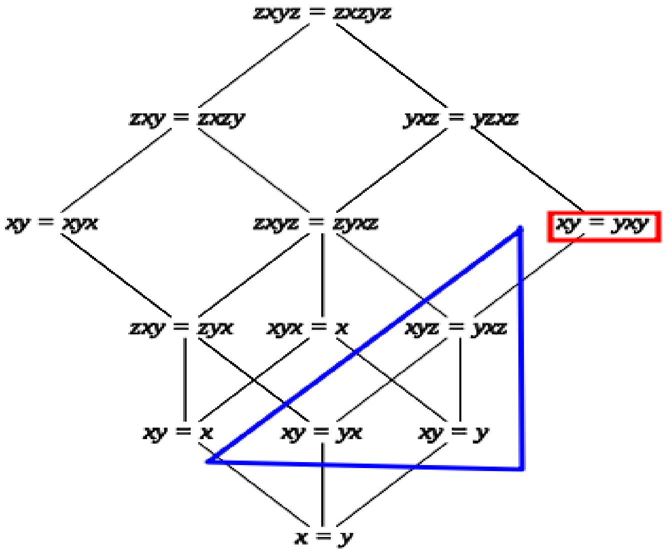

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$xy = y$

Consider the underlying order of an algebra in this variety. We have that $x \leq y$ iff $x = x \cdot y = y$. Hence the partial order is equality, and therefore the underlying orders of algebras in this variety are antichains.

Right-normal bands

Question: Which posets arise as underlying posets of algebras belonging to the subvariety determined by the identity $xyz = yxz$?

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Note also that this subvariety contains the subvarieties of semilattices and antichains. This fact will be fundamental for the characterization of normal posets that we present next.

The Projection onto the Quotient Semilattice

In the case where A is a right-normal band, the canonical projection to its quotient semilattice A/θ has a very nice property as an order homomorphism.

The Projection onto the Quotient Semilattice

Lemma

Let A be a right-normal band, θ the semilattice congruence of A, and $\pi : A \to A/\theta$ the canonical projection. For every $a \in A$, $\pi|_{a\downarrow}$ is an (order) isomorphism from $a\downarrow$ onto $\pi(a)\downarrow$.

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Let $a \in A$. We show that $\pi|_{a\downarrow}$ is an isomorphism of right-regular bands and therefore also an order isomorphism.

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Let $a \in A$. We show that $\pi|_{a\downarrow}$ is an isomorphism of right-regular bands and therefore also an order isomorphism. To prove injectivity, let $x, y \leq a$. If $\pi(x) = \pi(y)$, then $x \cdot y = y$ and $y \cdot x = x$. Hence

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$$x = x \cdot a = y \cdot x \cdot a = x \cdot y \cdot a = y \cdot a = y,$$

so $\pi|_{a\downarrow}$ is injective. To prove surjectivity, if $\pi(x) \leq \pi(a)$, then

$$\pi(x \cdot a) = \pi(x) \wedge \pi(a) = \pi(x),$$

and since $x \cdot a \in a\downarrow$, it follows that $\pi|_{a\downarrow}$ is surjective. □

Corollary

Let $(P, \leq)$ be a normal poset. Then there exist a semilattice S and a surjective homomorphism $f : P \to S$ such that for every $p \in P$, $f|_{p\downarrow}$ is an isomorphism from $p\downarrow$ onto $f(p)\downarrow$.

The characterization of normal posets

Surprisingly, the converse of the previous corollary also holds. That is:

Main Theorem

Let P be a poset such that there exist a semilattice S and a surjective homomorphism $f : P \to S$ such that for every $p \in P$, $f|_{p\downarrow}$ is an isomorphism from $p\downarrow$ onto $f(p)\downarrow$. Then P is a normal poset.

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$$B := \{\langle x, p\rangle \in S \times A : x \leq f(p)\}.$$

Geometrically, B consists of all points below the graph of f. Since B is downward closed, it is a substructure of $S \times A$.

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The characterization of normal posets

Proof.

Define $h : B \to P$ by $h(\langle x, p \rangle) = (f|_{p\downarrow})^{-1}(x)$.

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Define $h : B \to P$ by $h(\langle x, p\rangle) = (f|_{p\downarrow})^{-1}(x)$. We have that $\ker h$ is a congruence on B and

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given by

$$\varphi([\langle x, m\rangle]) = h(\langle x, m\rangle)$$

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It is not hard to check that the quotient semilattice of $B/\ker h$ is isomorphic to S and that the canonical projection behaves exactly as f. As a consequence, we have that one can recover the whole algebraic structure of a right-normal band from its underlying partial order together with its semilattice congruence.

Examples

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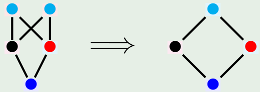

Here we exhibit a homomorphism $f : P \to S$ where the color of each node indicates its image under f. This homomorphism satisfies the conditions of the previous theorem, and therefore we may conclude that P is a normal poset.

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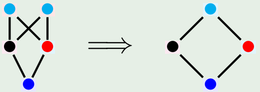

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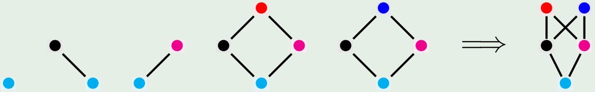

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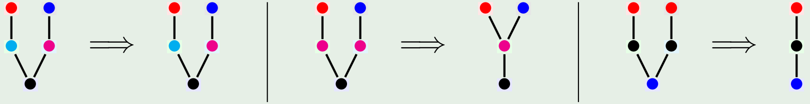

This example shows that, given a poset P, there might be different possible choices for the semilattice S and the homomorphism f.

Examples

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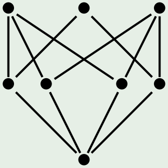

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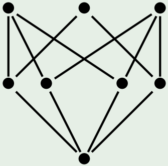

In this poset every principal ideal is a lower semilattice, but the poset is not normal.

Examples

We have seen that in the case of right-normal bands, the underlying partial order together with the congruence θ completely determine the product. This is not true for right-regular bands in general.

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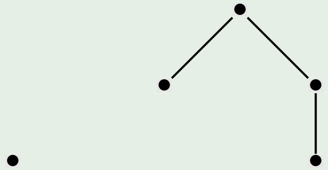

This poset admits two non-isomorphic right-regular band structures. The semilattice congruence is the same for both structures.

Trees

Definition

A *tree* T is a poset with a maximum element such that for every $x \in T$, $x\uparrow := \{y \in T : x \leq y\}$ is linearly ordered. Moreover, we say that a tree is *foliated* if for every $x \in T$ there exists at least one minimal element below x.

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Is AC necessary to define a right-normal band operation on every normal poset?

More precisely: if NP and RNB are the classes of normal posets and right-normal bands respectively, and $f : RNB \to NP$ is the map assigning to each normal band its underlying poset, is AC necessary to construct a right inverse for f?

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Thank you very much!