On the Dunn–Mortensen Result of Atomic Triviality

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Some Definitions

Theory

Let L be a logic with consequence relation $\vdash_L$. A theory T is a set of formulas closed under logical consequence with respect to $\vdash_L$. That is, if $\Gamma \subseteq T$ and $\Gamma \vdash_L \varphi$, then $\varphi \in T$. In this case, we say that L is the underlying logic of T.

Inconsistent Theory

A theory T is inconsistent if for some formula ψ, $\psi \in T$ and $\neg\psi \in T$.

Triviality Simpliciter

A theory T is trivial simpliciter if it contains every formula of the language S, i.e., $\varphi \in T$ for every $\varphi \in S$.

Paraconsistency

A logic L is paraconsistent if Explosion (from a contradiction, anything follows) is invalid, i.e., $\psi, \neg\psi \nvdash_L \varphi$.

The Paraconsistent Program

- One of the main goals of paraconsistent logics is to reason with inconsistencies. The invalidity of Explosion allows the existence of inconsistent but non-trivial theories.
- A mathematical theory T whose underlying logic L is paraconsistent can be inconsistent without being trivial.
- Suppose we have an inconsistent mathematical theory and that triviality can be derived from the inconsistency without using Explosion. This would be problematic for the paraconsistent program, since the invalidity of Explosion would no longer be sufficient to guarantee the non-triviality of inconsistent theories. In that case, additional resources would be needed to avoid triviality.
- The Dunn–Mortensen result on atomic triviality is an example of this situation.

Atomic triviality result

Some properties of the real numbers

i. For every $x \in \mathbb{R}$, there exists $y \in \mathbb{R}$ such that $x - y = 0$.

ii. For every $x \in \mathbb{R}$, $x * 1 = x$.

iii. If $x \neq 0$, then there exists $y \in \mathbb{R}$ such that $x * y = 1$.

iv. If $x, y \in \mathbb{R}$, then $x + y \in \mathbb{R}$.

v. For every $x \in \mathbb{R}$, $x * 0 = 0$.

vi. For every $x \in \mathbb{R}$, $x + 0 = x$.

vii. For every $x, y \in \mathbb{R}$, if $x = y$, then $x - y = 0$.

viii. For every $x, y \in \mathbb{R}$, if $x \neq y$, then $x - y \neq 0$.

Atomic triviality result

Equality Properties

- Symmetry: For all x, y, if $x = y$, then $y = x$.
- Transitivity: For all x, y, z, if $x = y$ and $y = z$, then $x = z$.
- Substitutivity of Identicals: For all x, y, if $x = y$ and $P(x)$, then $P(y)$.
- Replacement: For all x, y, if $x = y$, then $f(x) = f(y)$, where f can be $+$ or $*$.

Inference Rules

- *Modus Ponens* (MP)
- Universal Instantiation (UI)

Atomic Triviality Proof

Theorem 1

If $\boldsymbol{a = b}$, where a and b are distinct real numbers, is added to the theory of the real numbers, then every equation $w = r$ becomes true.

Proof:

1.	$a, b \in \mathbb{R}$	(Assumption)
2.	$a = b$	(Assumption)
3.	$a \neq b$	(Assumption)
4.	If $a \neq b$, then $a - b \neq 0$.	(viii, UI)
5.	$a - b \neq 0$	(3, 4, MP)
6.	$(a - b) \in \mathbb{R}$	(1, iv, MP)
7.	$(a - b) * \frac{1}{a-b} = 1$	(5, 6, iii, MP)
8.	$(a - b) * \frac{1}{a-b} = \frac{a-b}{a-b}$	(Product of fractions)
9.	$\frac{a-b}{a-b} = 1$	(7, 8, Transitivity)

Atomic Triviality Proof

10.	If $a = b$, then $a - b = 0$	(vii, UI)
11.	$a - b = 0$	(2, 10, MP)
12.	$0 * \frac{1}{a-b} = \frac{a-b}{a-b}$	(8, 11, Substitutivity of identicals)
13.	$0 * \frac{1}{a-b} = 0$	(v, UI)
14.	$0 = 0 * \frac{1}{a-b}$	(13, Symmetry)
15.	$0 = \frac{a-b}{a-b}$	(12, 14, Transitivity)
16.	$0 = 1$	(9, 15, Transitivity)
17.	$w * 0 = w * 1$	(16, Replacement)

Atomic Triviality Proof

18.	$w * 0 = 0$	(v, UI)
19.	$w * 1 = w$	(ii, UI)
20.	$0 = w * 1$	(17, 18, Substitutivity of identicals)
21.	$0 = w$	(19, 20, Transitivity)
22.	$r * 0 = r * 1$	(16, Replacement)
23.	$r * 0 = 0$	(v, UI)
24.	$r * 1 = r$	(ii, UI)
25.	$0 = r * 1$	(22, 23, Substitutivity of identicals)
26.	$0 = r$	(24, 25, Transitivity)
27.	$w = 0$	(21, Symmetry)
28.	$w = r$, for all $w, r \in \mathbb{R}$	(26, 27, Transitivity)

Functionality, Atomic Triviality, and the Catastrophe for Mathematics

- A paraconsistent logician could object that Theorem 1 does not yield triviality simpliciter. Nevertheless, for Mortensen, the result is still catastrophic.

Atomic Triviality

A theory T is atomically trivial if it contains every atomic formula of its language.

- For Mortensen (2009), atomic triviality is as undesirable as triviality simpliciter for mathematical theories, because mathematical reasoning is tied to functionality, i.e., to the substitutivity of identicals in atomic sentences.
- Thus, if all atomic formulas, such as equations, become true, then calculations become meaningless.

Inconsistent Equations in Fields and *RM3* (mod p)

- A natural way to describe the Dunn–Mortensen result is given by Mangraviti (2023): "The Dunn–Mortensen problem is that any inconsistent equation in a field will lead (...) to triviality" (p. 34).
- However, we can find a counterexample in the model *RM3* (mod p), where p is prime (Mortensen, 2009, p. 638).
- The model *RM3* (mod p) has as its domain the integers modulo p,

$$\mathbb{Z}_p = \{[0], [1], \ldots, [p-1]\},$$

which form a field with respect to addition and multiplication modulo p:

$$\langle \mathbb{Z}_p, + \pmod{p}, * \pmod{p} \rangle.$$

Its underlying logic is *RM3*, a paraconsistent logic with truth values T, B, and F. The valuation v of atomic formulas is as follows:

Inconsistent Equations in Fields and $RM3$ (mod p)

$$v(a = b) = B \quad \text{iff} \quad a \equiv b \pmod{p},$$
$$v(a = b) = F \quad \text{iff} \quad a \not\equiv b \pmod{p}.$$

- The designated truth values in $RM3$ are T and B. Thus, formulas whose value is T or B belong to the theory associated with the model.
- The evaluation conditions for negation in $RM3$ are:
  $v(\sim A) = T \quad \text{iff} \quad v(A) = F,$
  $v(\sim A) = B \quad \text{iff} \quad v(A) = B,$
  $v(\sim A) = F \quad \text{iff} \quad v(A) = T.$
- Hence, if $v(a = b) = B$, then both $a = b$ and $\sim (a = b)$ are designated in $RM3$ (mod p).

Does the Dunn–Mortensen Proof Go Through in *RM3* (mod p)?

- As we saw above, the multiplicative inverse property (iii) plays an important role in the proof.
- Does an inconsistent equation in *RM3* (mod p) provide a multiplicative inverse?

No. In *RM3* (mod p), we do not add a false classical equation to the theory. Property (iii) requires $a - b$ to be non-zero in the field, i.e. $[a - b] \neq [0]$. But in *RM3* (mod p), we have the following chain of evaluations:

$$v(a = b) = B \Rightarrow v(a - b = 0) = B \Rightarrow v(\sim (a - b = 0)) = B.$$

Although $\sim (a - b = 0)$ is designated, $[a - b]$ is not distinct from $[0]$ in $\mathbb{Z}_p$. Indeed,

$$v(a - b = 0) = B \quad \Longleftrightarrow \quad a - b \equiv 0 \pmod{p}.$$

Thus, $[a - b]$ and $[0]$ are the same object in the integers modulo p. However, if we add $a = b$ to *RM3* (mod p), where $a \not\equiv b \pmod{p}$, i.e. $\boldsymbol{v(a = b) = F}$, then the Dunn–Mortensen proof can go through, and atomic triviality follows.

Other Proofs: Some Definitions and Lemmas

Definition 1. (Divisibility)

Let $d, n \in \mathbb{Z}$ with $d \neq 0$. We say that d divides n, written $d \mid n$, if there exists $c \in \mathbb{Z}$ such that $n = d * c$.

Division Algorithm

If $n, d \in \mathbb{Z}$ with $d > 0$, then there exist unique integers q, r such that

$$n = d * q + r \quad \text{with} \quad 0 \leq r < d.$$

Lemma 1

If $d \mid n$, with $d > 0$, then $r = 0$ when we apply the division algorithm to n and d.

Lemma 2

If $d \nmid n$, with $d > 0$, then $r \neq 0$ when we apply the division algorithm to n and d.

Other Proofs: Some Definitions and Lemmas

Greatest Common Divisor

The greatest common divisor of $a, b \in \mathbb{Z}$, with a and b not both zero, is the greatest positive integer that divides both a and b. It is denoted by $\gcd(a, b)$.

Definition 2. (Coprime Integers)

Let $a, b \in \mathbb{Z}$, with a and b not both zero. We say that a and b are coprime if $\gcd(a, b) = 1$. I will write $a \star b$ to denote that a and b are coprime.

Other Proofs: Divisibility

Theorem 2

Let $a \in \mathbb{Z} \subset \mathbb{R}$. If $\boldsymbol{a \nmid a}$, then $w = r$ for all $w, r \in \mathbb{R}$.

Proof:

1. $a \nmid a$ (Assumption)
2. $a * 1 = a$ (ii)
3. $a \mid a$ (2, Def. 1)
4. $r_1 \neq 0$ (1, Lemma 2)
5. $r_2 = 0$ (3, Lemma 1)
6. $r_1 = r_2$ (Uniqueness of r by the division algorithm)
7. $r_1 = 0$ (5, 6, Substitutivity of identicals)
8. $r_1 * \frac{1}{r_1} = 1$ (4, iii)
9. $r_1 * \frac{1}{r_1} = 0 * \frac{1}{r_1}$ (7, Replacement)
10. $0 = 1$ (8, 9, v, Transitivity)

Other Proofs: Coprime Integers

Theorem 3

Let $a \in \mathbb{Z} \subset \mathbb{R}$. If $\boldsymbol{a \star a}$ with $a > 1$, then $w = r$ for all $w, r \in \mathbb{R}$.

Proof:

1.	$a \star a$	(Assumption)
2.	$a > 1$	(Assumption)
3.	$\gcd(a, a) = 1$	(1, Def. 2)
4.	$\gcd(a, a) = a$	(2)
5.	$a = 1$	(3, 4, Transitivity)
6.	$a - 1 = 0$	(5, Replacement)
7.	$a - 1 > 0$	(2)
8.	$(a-1) * \frac{1}{a-1} = 1$	(7, iii)
9.	$(a-1) * \frac{1}{a-1} = 0 * \frac{1}{a-1}$	(6, Replacement)
10.	$0 = 1$	(8, 9, v, Transitivity)

Weber's Proposal

- In order to avoid the atomic triviality of Theorems 1–3 and reason in an inconsistent mathematical context, one option is to invalidate some properties of equality, such as Transitivity or Substitutivity of Identicals.
- Another option is to modify some algebraic properties, especially the properties of additive and multiplicative inverses. Weber proposes that each x has an intrinsic zero 0_x and an intrinsic unit 1_x, satisfying:

$$x + (-x) = 0_x \qquad \text{and} \qquad x * x^{-1} = 1_x.$$

- Groups $\langle G, * \rangle$ satisfying $x * x^{-1} = e_x$ are called non-classical groups by Weber. When $e_x = e$ for all $x \in G$, we have a classical group.
- These groups have some important properties. For example, cancellation laws may fail. Also, intrinsic zeros 0_x have specific rules governing their behavior (Weber, 2021, pp. 221–226, pp. 232–236).

Conclusions

- We saw that there are fields in which an inconsistent equation is present, but atomic triviality does not follow.
- The proofs of Theorems 1, 2, and 3 have a common structure: they start **from a false atomic formula** involving a binary relation, namely $=$, $|$, and $\star$.
- The proofs of Theorems 1, 2, and 3 require the use of additive or multiplicative inverses.
- Finally, we examined Weber's proposal, which I take to be the most promising because it advocates for the study of inconsistent objects in their own right, rather than treating them merely as anomalies.

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