Semilattice Semantics for Logics of Evidence and Truth

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XXI SLALM
Bogotá, Colombia, Junio 1–5, 2026

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- Evidence is considered conclusive only when it is incontrovertible, and so strong that it leaves no room for reasonable doubt or contradiction. In LETs, however, evidence is not necessarily conclusive.
- There may exist conflicting pieces of evidence that lead us to accept a proposition and its negation, without leading to deductive collapse. This justifies the paraconsistency of LETs.

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The basic logic of evidence (BLE):

$$\frac{A \quad B}{A \wedge B}(I_\wedge) \qquad \frac{A \wedge B}{A}(E1_\wedge) \quad \frac{A \wedge B}{B}(E2_\wedge)$$

$$\frac{A}{A \vee B}(I1_\vee) \qquad \begin{array}{ccc} [A] & & [B] \\ \vdots & & \vdots \\ A \vee B & C & C \end{array} \over C (E_\vee)$$

$$\frac{B}{A \vee B}(I2_\vee)$$

$$\begin{array}{c} [A] \\ \vdots \\ B \end{array} \over A \to B (I_\to) \qquad \frac{A \to B \quad A}{B}(E_\to)$$

Logics of Evidence and Truth

$$\frac{\sim A}{\sim(A \wedge B)}(I1_{\sim\wedge}) \qquad \frac{\sim B}{\sim(A \wedge B)}(I2_{\sim\wedge})$$

$$\frac{\begin{array}{ccc}[\sim A] & [\sim B] & \\ \vdots & \vdots & \\ \sim(A \wedge B) & C & C\end{array}}{C}(E_{\sim\wedge})$$

$$\frac{\sim A \quad \sim B}{\sim(A \vee B)}(I_{\sim\vee}) \qquad \frac{\sim(A \vee B)}{\sim A}(E1_{\sim\vee}) \qquad \frac{\sim(A \vee B)}{\sim B}(E2_{\sim\vee})$$

$$\frac{A \quad \sim B}{\sim(A \to B)}(I_{\sim\to}) \qquad \frac{\sim(A \to B)}{A}(E1_{\sim\to}) \qquad \frac{\sim(A \to B)}{\sim B}(E2_{\sim\to})$$

$$\frac{A}{\sim\sim A}(I_{\sim\sim}) \qquad \frac{\sim\sim A}{A}(E_{\sim\sim})$$

Logics of Evidence and Truth

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Although defined from a different perspective, BLE turns out to be equivalent to Nelson's paraconsistent logic N4.

Logics of Evidence and Truth

The logic of evidence and truth $\mathrm{LET}_\jmath$: BLE + classicality operator.

$$\frac{\circ A \quad A \quad \sim A}{B}(EXP_\circ) \qquad \frac{\circ A \quad \begin{array}{c}[A]\\ \vdots \\ B\end{array} \quad \begin{array}{c}[\sim A]\\ \vdots \\ B\end{array}}{B}(PEM_\circ)$$

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- $\circ A$ expresses that A behaves classically. In terms of evidence, there is conclusive evidence about the truth or the falsity of A.

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- $A \wedge \circ A$ expresses that A is (indisputably) true.

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- $\circ A$ expresses that A behaves classically. In terms of evidence, there is conclusive evidence about the truth or the falsity of A.
- $A \wedge \circ A$ expresses that A is (indisputably) true.
- $\sim A \wedge \circ A$ expresses that A is (indisputably) false.

Logics of Evidence and Truth

The constructive logic of evidence and truth LET_C (Agudelo-Agudelo and Carnielli (2025)): BLE + co-implication + bottom + top.

$$\frac{\sim A \quad B}{B \prec A}(I_\prec) \qquad \frac{B \prec A}{\sim A}(E1_\prec) \quad \frac{B \prec A}{B}(E2_\prec)$$

$$\begin{array}{c}[\sim A] \\ \vdots \\ \sim B \end{array} \over \sim(B \prec A)(I_{\sim\prec}) \qquad \frac{\sim A \quad \sim(B \prec A)}{\sim B}(E_{\sim\prec})$$

$$\frac{}{\top}(I_\top) \qquad \frac{\bot}{A}(E_\bot)$$

$$\frac{}{\sim\bot}(I_{\sim\bot}) \qquad \frac{\sim\top}{A}(E_{\sim\top})$$

Logics of Evidence and Truth

In $\mathsf{LET}_{\mathsf{C}}$:

- Connectives of intuitionistic negation ($\neg$) and co-intuitionistic negation ($\multimap$) are defined by:

$$\neg A \stackrel{\text{def}}{=} A \to \bot, \qquad \multimap A \stackrel{\text{def}}{=} \top \prec A.$$

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- Connectives of consistency ($\circ$), inconsistency ($\bullet$), determinedness ($\star$), and undeterminedness ($\star$) are defined by:

$$\circ A = \neg(A \wedge \sim A), \qquad \bullet A = \multimap(A \vee \sim A),$$
$$\star A = \multimap(A \wedge \sim A), \qquad \star A = \neg(A \vee \sim A).$$

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$$\star A = \multimap(A \wedge \sim A), \qquad \bigstar A = \neg(A \vee \sim A).$$

- An equivalence connective $\leftrightarrow$ can be defined as usual by $A \leftrightarrow B \stackrel{\text{def}}{=} (A \to B) \wedge (B \to A)$, but substitution by equivalent formulas is not valid. However, a strong equivalence connective can be defined by $A \Leftrightarrow B \stackrel{\text{def}}{=} (A \leftrightarrow B) \wedge (\sim A \leftrightarrow \sim B)$, and substitution by strongly equivalent formulas is valid.

Logics of Evidence and Truth

- $\vdash_{\mathsf{LET_C}} (\sim \circ A) \Leftrightarrow (\bullet A)$, $\vdash_{\mathsf{LET_C}} (\sim \star A) \Leftrightarrow (\bigstar A)$,
  $\vdash_{\mathsf{LET_C}} (\bullet A) \leftrightarrow (A \wedge \sim A)$ (but $\nvdash_{\mathsf{LET_C}} (\bullet A) \Leftrightarrow (A \wedge \sim A)$), and
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- Unlike $\mathsf{LET_J}$, where the classicality connective allows one to simultaneously recover explosiveness and determinedness, in $\mathsf{LET_C}$ these properties can be recovered independently by means of the connectives $\circ$ and $\bigstar$, respectively.

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- Unlike $\mathsf{LET_J}$, where the classicality connective allows one to simultaneously recover explosiveness and determinedness, in $\mathsf{LET_C}$ these properties can be recovered independently by means of the connectives $\circ$ and $\star$, respectively.
- BLE and $\mathsf{LET_J}$ are equipped with valuation semantics and Kripke-style semantics (Antunes et al., 2020), and $\mathsf{LET_C}$ is equipped with a Kripke-style semantics and a realisability interpretation (Agudelo-Agudelo and Carnielli, 2025). However, none of these semantics contains elements that explicitly represent evidence.

Semilattice Semantics

The semilattice semantics proposed by Urquhart (1971) (see also (Urquhart, 1972, 1973)) provides a model for interpreting logical systems—particularly relevant logics—by means of the structure of a join-semilattice with identity (an algebraic structure equipped with a binary operation, called join, which is commutative, associative, idempotent, and possesses an identity element).

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- If x and y represent pieces of information, the join $x \sqcup y$ represents the piece of information containing exactly the information present in either x or y, which intuitively accounts for the properties of commutativity, associativity, and idempotence.

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- The identity element corresponds to the empty piece of information.

Semilattice Semantics

- In the semilattice semantics, a valuation function assigns to each propositional variable p a set of elements of the semilattice: those that represent pieces of information that support the truth of p. This valuation is then extended inductively to all formulas under some intuitive conditions.

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- In the semilattice semantics, a valuation function assigns to each propositional variable p a set of elements of the semilattice: those that represent pieces of information that support the truth of p. This valuation is then extended inductively to all formulas under some intuitive conditions.
- The semilattice semantics was originally proposed to interpret relevance logics. Although Urquhart (1973) proposed modifications to interpret intuitionistic implication, a characterisation of propositional intuitionistic logic was obtained only recently by Weiss (2021).

Semilattice Semantics for Logics of Evidence and Truth

The main objective of this work is to characterise BLE, $\mathrm{LET}_{\mathsf{C}}$, and $\mathrm{LET}_{\mathsf{J}}$ by means of semilattice semantics. Since pieces of information can naturally be interpreted as evidence, these semantics adequately formalise the guiding idea behind the construction of these logics.

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In order to obtain a semilattice semantics for BLE, we first present a semilattice semantics for the implicative fragment of intuitionistic logic ($\mathsf{Int}_{\rightarrow}$). This semantics is then extended to the positive fragment of intuitionistic logic (Int^{+}), and later to BLE, $\mathsf{LET}_{\mathsf{C}}$, and $\mathsf{LET}_{\mathsf{J}}$.

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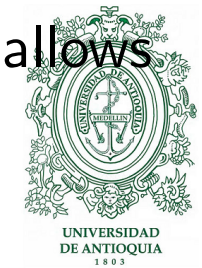
Although Weiss (2021) provides a semilattice semantics for the full propositional system of intuitionistic logic (Int), which can be restricted to the fragments $\mathrm{Int}_{\rightarrow}$ and Int^{+}, our proof of the adequacy (i.e. soundness and completeness) of these semantics differs from that of Weiss and allows us to extend these ideas to BLE, $\mathrm{LET}_{\mathsf{C}}$, and $\mathrm{LET}_{\mathsf{J}}$.



Semilattice Semantics for $\mathrm{Int}_{\rightarrow}$

Definition 3.1

A semilattice frame is a join-semilattice with identity, i.e. an algebraic structure $\langle S, \sqcup, 0\rangle$, where S is a set, $\sqcup$ is a binary operation on S, called join, that is associative, commutative, and idempotent, and 0 is an identity element.

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Definition 3.2

A semilattice model for $\mathsf{Int}_{\rightarrow}$ is a structure $\langle S, \sqcup, 0, v\rangle$, where $\langle S, \sqcup, 0\rangle$ is a semilattice frame and $v : V \rightarrow \mathbf{P}(S)$ (where $\mathbf{P}(S)$ denotes the power set of S) is a valuation function that satisfies the heredity condition:

$$\text{If } x \in v(p), \text{ then for all } y \in S \text{ it holds that } x \sqcup y \in v(p).$$

Semilattice Semantics for $\mathsf{Int}_{\rightarrow}$

Definition 3.3

Given a semilattice model $\mathcal{M} = \langle S, \sqcup, 0, v \rangle$ for $\mathsf{Int}_{\rightarrow}$, a verification relation $\Vdash_{\mathcal{M}}$ between elements of S and formulas of $\mathsf{Int}_{\rightarrow}$ is inductively defined by:

$$x \Vdash_{\mathcal{M}} p \text{ iff } x \in v(p) \text{ (for } p \in V),$$
$$x \Vdash_{\mathcal{M}} A \rightarrow B \text{ iff for all } y \in S \text{ it holds that } y \nVdash_{\mathcal{M}} A \text{ or } x \sqcup y \Vdash_{\mathcal{M}} B.$$

A formula $A \in \mathcal{F}_{\mathsf{R}_{\rightarrow}}$ is verified in $\mathcal{M}$ if $0 \Vdash_{\mathcal{M}} A$.

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For a logic $\mathsf{L} \in \{\mathsf{Int}_{\rightarrow}, \mathsf{Int}^{+}\}$, a formula $A \in \mathcal{F}_{\mathsf{L}}$ is valid in the semilattice semantics for L, denoted by $\Vdash_{\mathsf{L}} A$, if A is verified in all semilattice models for L.

Semilattice Semantics for $\mathsf{Int}_{\rightarrow}$

The proof of adequacy of the semilattice semantics for $\mathsf{Int}_{\rightarrow}$ is obtained through the following steps:

1. It is shown that semilattice models can be viewed as particular cases of relational (or Kripke-style) models for this logic.

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2. It is shown that relational models can be extended to semilattice models while preserving verifiability.

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1. It is shown that semilattice models can be viewed as particular cases of relational (or Kripke-style) models for this logic.
2. It is shown that relational models can be extended to semilattice models while preserving verifiability.
3. The previous facts, together with the adequacy of the relational semantics for $\mathrm{Int}_{\rightarrow}$, are used to prove the adequacy of the semilattice semantics.

Semilattice Semantics for $\mathrm{Int}_{\rightarrow}$

Definition 3.4

A relational frame is a partially ordered set (i.e. a structure $\langle S, \leqslant \rangle$, where S is a set and $\leqslant$ is a binary relation on S that is reflexive, transitive and antisymmetric). Elements of S are called states and $\leqslant$ is called the accessibility relation.

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Definition 3.5

A relational model for $\mathrm{Int}_{\rightarrow}$ is a structure $\langle S, \leqslant, v \rangle$, where $\langle S, \leqslant \rangle$ is a relational frame, and $v : V \rightarrow \mathbf{P}(S)$ is a valuation function satisfying the following heredity condition:

$$\text{If } x \in v(p), \text{ then for all } y \in S, \text{ if } x \leqslant y, \text{ then } y \in v(p).$$

Semilattice Semantics for $\mathsf{Int}_{\rightarrow}$

Definition 3.6

Given a relational model $\mathcal{M} = \langle S, \leqslant, v \rangle$ for $\mathsf{Int}_{\rightarrow}$, a verification relation $\models_{\mathcal{M}}$ between elements of S and formulas of $\mathsf{Int}_{\rightarrow}$ is inductively defined by:

$$x \models_{\mathcal{M}} p \text{ iff } x \in v(p) \text{ (for } p \in V);$$
$$x \models_{\mathcal{M}} A \rightarrow B \text{ iff for all } y \in S, \text{ if } x \leqslant y, \text{ then } y \not\models_{\mathcal{M}} A \text{ or } y \models_{\mathcal{M}} B.$$

A formula $A \in \mathcal{F}_{\mathsf{Int}_{\rightarrow}}$ is verified in $\mathcal{M}$ if for all $x \in S$ it holds that $x \models_{\mathcal{M}} A$.

Semilattice Semantics for $\mathsf{Int}_{\to}$

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Given a relational model $\mathcal{M} = \langle S, \leqslant, v \rangle$ for $\mathsf{Int}_{\to}$, a verification relation $\models_{\mathcal{M}}$ between elements of S and formulas of $\mathsf{Int}_{\to}$ is inductively defined by:

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For a logic $\mathsf{L} \in \{\mathsf{Int}_{\to}, \mathsf{Int}^{+}\}$, a formula $A \in \mathcal{F}_{\mathsf{L}}$ is valid in the relational semantics for L, denoted by $\models_{\mathsf{L}} A$, if A is verified in all relational models for L.

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For a logic $\mathsf{L} \in \{\mathsf{Int}_{\to}, \mathsf{Int}^{+}\}$, a formula $A \in \mathcal{F}_{\mathsf{L}}$ is valid in the relational semantics for L, denoted by $\models_{\mathsf{L}} A$, if A is verified in all relational models for L.

Theorem 3.7 (Adequacy of the relational semantics for $\mathsf{Int}_{\to}$)

Let $A \in \mathcal{F}_{\mathsf{Int}_{\to}}$. We have that $\vdash_{\mathsf{Int}_{\to}} A$ iff $\models_{\mathsf{Int}_{\to}} A$.

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Differences between the semilattice semantics and the relational semantics for $\mathrm{Int}_{\rightarrow}$:

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Differences between the semilattice semantics and the relational semantics for $\mathsf{Int}_{\rightarrow}$:

1. Semilattice frames are semilattices with identity, whereas relational frames are partial orders.
2. The condition for the verification of implication in the two semantics is different.
3. A formula is verified in a semilattice model if it is verified at the identity element, whereas a formula is verified in a relational model if it is verified at all states.

Semilattice Semantics for $\mathrm{Int}_{\rightarrow}$

A join-semilattice can be alternatively defined as follows.

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Lemma 3.9

Let $\langle S, \sqcup \rangle$ be a join-semilattice regarded as an algebraic structure. Then the structure $\langle S, \leqslant \rangle$, where $x \leqslant y$ iff $x \sqcup y = y$, forms a join-semilattice when viewed as an ordered set. Moreover, if $\langle S, \sqcup \rangle$ has an identity element 0, then 0 is the least element of $\langle S, \leqslant \rangle$.

Semilattice Semantics for $\mathrm{Int}_{\rightarrow}$

Lemma 3.10

Let $\langle S, \leqslant \rangle$ be a join-semilattice viewed as an ordered set. Then the structure $\langle S, \sqcup \rangle$, where $x \sqcup y = \sup\{x, y\}$, forms a join-semilattice regarded as an algebraic structure. Moreover, if $\langle S, \leqslant \rangle$ has a least element 0, then 0 is an identity element of $\langle S, \sqcup \rangle$.

Semilattice Semantics for $\mathsf{Int}_{\rightarrow}$

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Let $\langle S, \leqslant \rangle$ be a join-semilattice viewed as an ordered set. Then the structure $\langle S, \sqcup \rangle$, where $x \sqcup y = \sup\{x, y\}$, forms a join-semilattice regarded as an algebraic structure. Moreover, if $\langle S, \leqslant \rangle$ has a least element 0, then 0 is an identity element of $\langle S, \sqcup \rangle$.

Lemma 3.11

Let $\langle S, \sqcup, 0, v \rangle$ be a semilattice model for $\mathsf{Int}_{\rightarrow}$, and let $\leqslant$ be the binary relation on S defined by $x \leqslant y$ iff $x \sqcup y = y$. For any $x \in S$, the following conditions are equivalent.

1. *For all $y \in S$, it holds that $y \nVdash_{\mathsf{Int}_{\rightarrow}} A$ or $x \sqcup y \Vdash_{\mathsf{Int}_{\rightarrow}} B$.*
2. *For all $y \in S$, if $x \leqslant y$, then $y \nVdash_{\mathsf{Int}_{\rightarrow}} A$ or $y \Vdash_{\mathsf{Int}_{\rightarrow}} B$.*

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Semilattice Semantics for $\mathrm{Int}_{\rightarrow}$

Definition 3.12

Let $\mathcal{S} = \langle S, \leqslant_S \rangle$ and $\mathcal{P} = \langle P, \leqslant_P \rangle$ be partially ordered sets. A function $f : S \rightarrow P$ is an order-embedding of $\mathcal{S}$ into $\mathcal{P}$ if f is one-to-one and $\forall x, y \in S(x \leqslant_S y \leftrightarrow f(x) \leqslant_P f(y))$.

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Definition 3.13

A complete lattice is a partially ordered set $\mathcal{S} = \langle S, \leqslant \rangle$ in which every subset of S has sumpremum and infimum.

Semilattice Semantics for $\mathrm{Int}_{\rightarrow}$

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Definition 3.13

A complete lattice is a partially ordered set $\mathcal{S} = \langle S, \leqslant \rangle$ in which every subset of S has sumpremum and infimum.

Definition 3.14

Let $\mathcal{S} = \langle S, \leqslant_S \rangle$ be a partially ordered set and $\mathcal{P} = \langle P, \leqslant_P \rangle$ be a complete lattice. Then $\mathcal{P}$ is a completion of $\mathcal{S}$ if there is an order-embedding of $\mathcal{S}$ into $\mathcal{P}$.

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Semilattice Semantics for $\mathrm{Int}_{\rightarrow}$

Definition 3.15

Let $\mathcal{S} = \langle S, \leqslant \rangle$ be a partially ordered set. A set P is a down-set of $\mathcal{S}$ if $P \subseteq S$, and for all $x \in P$ and $y \in S$, if $y \leqslant x$, then $y \in P$. The set of down-sets of S is denoted by $\mathcal{O}(S)$.

Semilattice Semantics for $\mathrm{Int}_{\rightarrow}$

Definition 3.15

Let $\mathcal{S} = \langle S, \leqslant \rangle$ be a partially ordered set. A set P is a down-set of $\mathcal{S}$ if $P \subseteq S$, and for all $x \in P$ and $y \in S$, if $y \leqslant x$, then $y \in P$. The set of down-sets of S is denoted by $\mathcal{O}(S)$.

Lemma 3.16

Let $\mathcal{S} = (S, \leqslant)$ be a partially ordered set, and let $\mathcal{S}_c = \langle \mathcal{O}(S), \subseteq \rangle$.

1. *$\mathcal{S}_c$ is a complete lattice, and $\varnothing$ is its less element.*
2. *$\mathcal{S}_c$ is a completion of $\mathcal{S}$, and $f : S \to \mathcal{O}(S)$ defined by $f(x) = {\downarrow} x$, where ${\downarrow} x = \{y \in S \mid y \leqslant x\}$, is the canonical order-embedding of $\mathcal{S}$ into $\mathcal{S}_c$.*

Semilattice Semantics for $\mathsf{Int}_{\rightarrow}$

Definition 3.17

Let $\mathcal{M} = \langle S, \leqslant, v \rangle$ be a relational model for $\mathsf{Int}_{\rightarrow}$. The completion of $\mathcal{M}$ is the structure $\mathcal{M}_c = \langle \mathcal{O}(S), \subseteq, v_c \rangle$, where $v_c : V \rightarrow \mathbf{P}(\mathcal{O}(S))$ is defined by:

$$v_c(p) = \mathcal{O}(S) \backslash \{Y \in \mathcal{O}(S) \mid Y \subseteq {\downarrow} x \text{ and } x \notin v(p), \text{ for some } x \in S\}.$$

Semilattice Semantics for $\mathsf{Int}_{\rightarrow}$

Definition 3.17

Let $\mathcal{M} = \langle S, \leqslant, v \rangle$ be a relational model for $\mathsf{Int}_{\rightarrow}$. The completion of $\mathcal{M}$ is the structure $\mathcal{M}_c = \langle \mathcal{O}(S), \subseteq, v_c \rangle$, where $v_c : V \rightarrow \mathbf{P}(\mathcal{O}(S))$ is defined by:

$$v_c(p) = \mathcal{O}(S) \backslash \{Y \in \mathcal{O}(S) \mid Y \subseteq {\downarrow} x \text{ and } x \notin v(p), \text{ for some } x \in S\}.$$

Lemma 3.18

Let $\mathcal{M} = \langle S, \leqslant, v \rangle$ be a relational model for $\mathsf{Int}_{\rightarrow}$. Its completion satisfies the following properties:

1. *It is a relational model for $\mathsf{Int}_{\rightarrow}$.*
2. *For every $x \in S$ and $A \in \mathcal{F}_{\mathsf{Int}_{\rightarrow}}$, we have that ${\downarrow} x \Vdash_{\mathcal{M}_c} A$ iff $x \Vdash_{\mathcal{M}} A$.*

Semilattice Semantics for $\mathsf{Int}_{\rightarrow}$

Theorem 3.19

Let $A \in \mathcal{F}_{\mathsf{Int}_{\rightarrow}}$. We have that $\Vdash_{\mathsf{Int}_{\rightarrow}} A$ iff $\models_{\mathsf{Int}_{\rightarrow}} A$.

Semilattice Semantics for $\mathsf{Int}_{\rightarrow}$

Theorem 3.19

Let $A \in \mathcal{F}_{\mathsf{Int}_{\rightarrow}}$. We have that $\Vdash_{\mathsf{Int}_{\rightarrow}} A$ iff $\models_{\mathsf{Int}_{\rightarrow}} A$.

Theorem 3.20 (Adequacy of the semilattice semantics for $\mathsf{Int}_{\rightarrow}$)

Let $A \in \mathcal{F}_{\mathsf{Int}_{\rightarrow}}$. We have that $\vdash_{\mathsf{Int}_{\rightarrow}} A$ iff $\Vdash_{\mathsf{Int}_{\rightarrow}} A$.

Semilattice Semantics for Int^+

The semilattice semantics for $\mathsf{Int}_{\rightarrow}$ can be easily and naturally extended to Int^+.

Definition 3.21

A semilattice model for Int^+ is defined by adding to the definition of semilattice model for $\mathsf{Int}_{\rightarrow}$ the following conditions to the verification relation:

$$x \Vdash_{\mathcal{M}} A \wedge B \text{ iff } x \Vdash_{\mathcal{M}} A \text{ and } x \Vdash_{\mathcal{M}} B,$$
$$x \Vdash_{\mathcal{M}} A \vee B \text{ iff } x \Vdash_{\mathcal{M}} A \text{ or } x \Vdash_{\mathcal{M}} B.$$

Semilattice Semantics for Int^+

The semilattice semantics for $\mathsf{Int}_{\rightarrow}$ can be easily and naturally extended to Int^+.

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A semilattice model for Int^+ is defined by adding to the definition of semilattice model for $\mathsf{Int}_{\rightarrow}$ the following conditions to the verification relation:

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$$x \Vdash_{\mathcal{M}} A \vee B \text{ iff } x \Vdash_{\mathcal{M}} A \text{ or } x \Vdash_{\mathcal{M}} B.$$

The proof of the adequacy of the semilattice semantics for Int^+ is obtained by straightforward extensions of the previous results.

Theorem 3.22 (Adequacy of the semilattice semantics for Int^+)

Let $A \in \mathcal{F}_{\mathsf{Int}^+}$. We have that $\vdash_{\mathsf{Int}^+} A$ iff $\Vdash_{\mathsf{Int}^+} A$.

Semilattice Semantics for Int^+

At first glance, it may seem that the semilattice semantics for Int^+ can also be easily and naturally extended to the full system of Int by adding the following apparently obvious clause for the bottom constant:

$$x \nVdash_{\mathcal{M}} \bot.$$

Semilattice Semantics for Int^+

At first glance, it may seem that the semilattice semantics for Int^+ can also be easily and naturally extended to the full system of Int by adding the following apparently obvious clause for the bottom constant:

$$x \nVdash_{\mathcal{M}} \bot.$$

However, as shown by Weiss (2021, Section 4), this condition actually yields an adequate semantics for Jankov's logic KC, a proper deductive extension of Int in which the weak excluded middle schema $\neg A \vee \neg \neg A$ holds.

Semilattice Semantics for Int^+

In the relational semantics for Int, it is possible to have models $\mathcal{M} = \langle S, \leqslant, v \rangle$ with mutually $\neg$-contradictory states—i.e., states $x, y \in S$ such that $x \vDash_{\mathcal{M}} A$ and $y \vDash_{\mathcal{M}} \neg A$ for some $A \in \mathcal{F}_{\mathsf{Int}}$. If such models are completed according to Definition 3.17, taking into account the usual conditions of the verification relation $\vDash_{\mathcal{M}}$ for $\rightarrow$, $\wedge$, $\vee$, and $\bot$, and the heredity property, the resulting model $\mathcal{M}_c$ would contain contradictory states; i.e. states $X \in \mathcal{O}(S)$ such that $X \vDash_{\mathcal{M}_c} A$ and $X \vDash_{\mathcal{M}_c} \neg A$, which is not allowed, since from $X \vDash_{\mathcal{M}_c} \neg A$ it follows that $X \nvDash_{\mathcal{M}_c} A$, yielding a meta-logical contradiction.

Semilattice Semantics for Int^+

Weiss's strategy for extending the semilattice semantics to Int consists in extending the domain of the valuation function v in semilattice models to $V \cup \{\bot\}$, adding the following condition:

$$x \in v(\bot) \text{ implies } x \in v(p) \text{ for all } p \in V.$$

Semilattice Semantics for $\mathbf{Int}^+$

Weiss's strategy for extending the semilattice semantics to Int consists in extending the domain of the valuation function v in semilattice models to $V \cup \{\bot\}$, adding the following condition:

$$x \in v(\bot) \text{ implies } x \in v(p) \text{ for all } p \in V.$$

To establish completeness of the semilattice semantics for KC, Weiss appeals to a property of extensions of Int, rather than to the method of completing relational refutational models to obtain a semilattice refutational model.

Semilattice Semantics for BLE

Now, we will extend the previous results to BLE.

Definition 3.23

A semilattice model for BLE is a structure $\langle S, \sqcup, 0, v^+, v^- \rangle$, where $\langle S, \sqcup, 0 \rangle$ is a semilattice frame, and $v^+ : V \to \mathbf{P}(S)$ and $v^- : V \to \mathbf{P}(S)$ are the verification and falsification valuation functions, respectively, and satisfy the following heredity condition:

$$\text{If } x \in v^*(p), \text{then for all } y \in S \text{ it holds that } x \sqcup y \in v^*(p).$$

Semilattice Semantics for BLE

Definition 3.24

Given a semilattice model $\mathcal{M} = \langle S, \sqcup, 0, v^+, v^- \rangle$ for BLE, the verification and falsification relations $\Vdash^+_{\mathcal{M}}$ and $\Vdash^-_{\mathcal{M}}$ are defined by:

$$
\begin{aligned}
x \Vdash^*_{\mathcal{M}} p &\text{ iff } x \in v^*(p) \text{ (for } p \in V),\\
x \Vdash^+_{\mathcal{M}} \sim A &\text{ iff } x \Vdash^-_{\mathcal{M}} A,\\
x \Vdash^+_{\mathcal{M}} A \wedge B &\text{ iff } x \Vdash^+_{\mathcal{M}} A \text{ and } x \Vdash^+_{\mathcal{M}} B,\\
x \Vdash^+_{\mathcal{M}} A \vee B &\text{ iff } x \Vdash^+_{\mathcal{M}} A \text{ or } x \Vdash^+_{\mathcal{M}} B,\\
x \Vdash^+_{\mathcal{M}} A \to B &\text{ iff for all } y \in S \text{ it holds that } y \nVdash^+_{\mathcal{M}} A \text{ or } x \sqcup y \Vdash^+_{\mathcal{M}} B,\\
x \Vdash^-_{\mathcal{M}} \sim A &\text{ iff } x \Vdash^+_{\mathcal{M}} A,\\
x \Vdash^-_{\mathcal{M}} A \wedge B &\text{ iff } x \Vdash^-_{\mathcal{M}} A \text{ or } x \Vdash^-_{\mathcal{M}} B,\\
x \Vdash^-_{\mathcal{M}} A \vee B &\text{ iff } x \Vdash^-_{\mathcal{M}} A \text{ and } x \Vdash^-_{\mathcal{M}} B,\\
x \Vdash^-_{\mathcal{M}} A \to B &\text{ iff } x \Vdash^+_{\mathcal{M}} A \text{ and } x \Vdash^-_{\mathcal{M}} B.
\end{aligned}
$$

A formula $A \in \mathcal{F}_{\text{BLE}}$ is verified (respectively, falsified) in $\mathcal{M}$ if $0 \Vdash^+_{\mathcal{M}} A$ (respectively, $0 \Vdash^-_{\mathcal{M}} A$).

Semilattice Semantics for BLE

Definition 3.25

A relational model for BLE is a structure $\langle S, \leqslant, v^+, v^- \rangle$, where $\langle S, \leqslant \rangle$ is a relational frame, $v^+ : V \to \mathbf{P}(S)$ and $v^- : V \to \mathbf{P}(S)$ are the verification and falsification valuation functions, respectively, and satisfy the following heredity condition:

$$\text{If } x \in v^*(p), \text{ then for all } y \in S, \text{ if } x \leqslant y, \text{ then } y \in v^*(p).$$

Semilattice Semantics for BLE

Definition 3.26

Given a relational model $\mathcal{M} = \langle S, \leqslant, v^+, v^- \rangle$ for BLE, the verification and falsification relations $\models^+_{\mathcal{M}}$ and $\models^-_{\mathcal{M}}$ are defined inductively by:

$$
\begin{aligned}
x \models^*_{\mathcal{M}} p &\text{ iff } x \in v^*(p) \text{ (for } p \in V);\\
x \models^+_{\mathcal{M}} \sim A &\text{ iff } x \models^-_{\mathcal{M}} A;\\
x \models^+_{\mathcal{M}} A \wedge B &\text{ iff } x \models^+_{\mathcal{M}} A \text{ and } x \models^+_{\mathcal{M}} B;\\
x \models^+_{\mathcal{M}} A \vee B &\text{ iff } x \models^+_{\mathcal{M}} A \text{ or } x \models^+_{\mathcal{M}} B;\\
x \models^+_{\mathcal{M}} A \to B &\text{ iff for all } y \in S, \text{ if } x \leqslant y, \text{ then } y \not\models^+_{\mathcal{M}} A \text{ or } y \models^+_{\mathcal{M}} B;\\
x \models^-_{\mathcal{M}} \sim A &\text{ iff } x \models^+_{\mathcal{M}} A;\\
x \models^-_{\mathcal{M}} A \wedge B &\text{ iff } x \models^-_{\mathcal{M}} A \text{ or } x \models^-_{\mathcal{M}} B;\\
x \models^-_{\mathcal{M}} A \vee B &\text{ iff } x \models^-_{\mathcal{M}} A \text{ and } x \models^-_{\mathcal{M}} B;\\
x \models^-_{\mathcal{M}} A \to B &\text{ iff } x \models^+_{\mathcal{M}} A \text{ and } x \models^-_{\mathcal{M}} B.
\end{aligned}
$$

A formula $A \in \mathcal{F}_{\text{BLE}}$ is verified (respectively, falsified) in $\mathcal{M}$, denoted by $\models^+_{\mathcal{M}} A$ (respectively, $\models^-_{\mathcal{M}} A$), if for all $x \in S$ it holds that $x \models^+_{\mathcal{M}} A$ (respectively, $x \models^-_{\mathcal{M}} A$).

Semilattice Semantics for BLE

Theorem 3.27 (Adequacy of the relational semantics for BLE)

Let $A \in \mathcal{F}_{\mathsf{BLE}}$. *We have that* $\vdash_{\mathsf{BLE}} A$ *iff* $\models_{\mathsf{Int}_{\rightarrow}} A$.

Semilattice Semantics for BLE

Theorem 3.27 (Adequacy of the relational semantics for BLE)

Let $A \in \mathcal{F}_{\mathsf{BLE}}$. We have that $\vdash_{\mathsf{BLE}} A$ iff $\models_{\mathsf{Int}_{\rightarrow}} A$.

Lemma 3.28

Let $\langle S, \sqcup, 0, v^+, v^- \rangle$ be a semilattice model for BLE*, and let $\leqslant$ be the binary relation on S defined by $x \leqslant y$ iff $x \sqcup y = y$. For any $x \in S$, the following conditions are equivalent.*

1. *For all $y \in S$, it holds that $y \nVdash^+_{\mathsf{BLE}} A$ or $x \sqcup y \Vdash^+_{\mathsf{BLE}} B$.*
2. *For all $y \in S$, if $x \leqslant y$, then $y \nVdash^+_{\mathsf{BLE}} A$ or $y \Vdash^+_{\mathsf{BLE}} B$.*

Semilattice Semantics for BLE

Definition 3.29

Let $\mathcal{M} = \langle S, \leqslant, v^+, v^- \rangle$ be a relational model for BLE. The completion of $\mathcal{M}$ is the structure $\mathcal{M}_c = \langle \mathcal{O}(S), \subseteq, v_c^+, v_c^- \rangle$, where $v_c^* : V \to \mathbf{P}(\mathcal{O}(S))$ is defined by:

$$v_c^*(p) = \mathcal{O}(S) \backslash \{Y \in \mathcal{O}(S) \mid Y \subseteq {\downarrow} x \text{ and } x \notin v^*(p), \text{ for some } x \in S\}.$$

Semilattice Semantics for BLE

Definition 3.29

Let $\mathcal{M} = \langle S, \leqslant, v^+, v^- \rangle$ be a relational model for BLE. The completion of $\mathcal{M}$ is the structure $\mathcal{M}_c = \langle \mathcal{O}(S), \subseteq, v_c^+, v_c^- \rangle$, where $v_c^* : V \to \mathbf{P}(\mathcal{O}(S))$ is defined by:

$$v_c^*(p) = \mathcal{O}(S) \backslash \{Y \in \mathcal{O}(S) \mid Y \subseteq {\downarrow} x \text{ and } x \notin v^*(p), \text{ for some } x \in S\}.$$

Lemma 3.30

Let $\mathcal{M} = \langle S, \leqslant, v^+, v^- \rangle$ be a relational model for BLE*. Its completion satisfies the following properties:*

1. *It is a relational model for* BLE*.*
2. *For every $x \in S$ and $A \in \mathcal{F}_{\text{BLE}}$, we have that ${\downarrow} x \models^*_{\mathcal{M}_c} A$ iff $x \models^*_{\mathcal{M}} A$.*

Semilattice Semantics for BLE

Theorem 3.31

Let $A \in \mathcal{F}_{\mathsf{BLE}}$. *We have that* $\Vdash_{\mathsf{BLE}} A$ *iff* $\models_{\mathsf{BLE}} A$.

Semilattice Semantics for BLE

Theorem 3.31

Let $A \in \mathcal{F}_{\mathsf{BLE}}$. *We have that* $\Vdash_{\mathsf{BLE}} A$ *iff* $\models_{\mathsf{BLE}} A$.

Theorem 3.32 (Adequacy of the semilattice semantics for BLE)

Let $A \in \mathcal{F}_{\mathsf{BLE}}$. *We have that* $\vdash_{\mathsf{BLE}} A$ *iff* $\Vdash_{\mathsf{BLE}} A$.

Semilattice Semantics for LET_C

The characterisation of LET_C by semilattice semantics requires much more work due to the interpretation of $\bot$.

Definition 3.33

A semilattice model for LET_C is a structure $\langle S, \sqcup, 0, v^+, v^- \rangle$, where $\langle S, \sqcup, 0 \rangle$ is a semilattice frame, and $v^+ : V \cup \{\bot, \top\} \to \mathbf{P}(S)$ and $v^- : V \cup \{\bot, \top\} \to \mathbf{P}(S)$ are the verification and falsification valuation functions, respectively. These functions must satisfy the following conditions:

$v^+(\top) = S$;

$v^-(\bot) = v^+(\top)$;

$v^+(\bot) = v^-(\top)$;

if $x \in v^+(\bot)$, then for all $p \in V$ and $* \in \{+, -\}$ it holds that $x \in v^*(p)$;

if $x \in v^*(a)$, then for all $y \in S$ it holds that $x \sqcup y \in v^*(a)$.

Semilattice Semantics for $\mathsf{LET}_{\mathsf{C}}$

Definition 3.34

Given a semilattice model $\mathcal{M} = \langle S, \sqcup, 0, v^+, v^- \rangle$ for $\mathsf{LET}_{\mathsf{C}}$, the verification and falsification relations $\Vdash^+_{\mathcal{M}}$ and $\Vdash^-_{\mathcal{M}}$ are defined inductively by adding the following clauses to those given in Definition 3.23:

$$x \Vdash^*_{\mathcal{M}} a \text{ iff } x \in v^*(a) \text{ (for } a \in \{\bot, \top\}),$$
$$x \Vdash^+_{\mathcal{M}} B \prec A \text{ iff } x \Vdash^+_{\mathcal{M}} B \text{ and } x \Vdash^-_{\mathcal{M}} A,$$
$$x \Vdash^-_{\mathcal{M}} B \prec A \text{ iff for all } y \in S \text{ it holds that } y \nVdash^-_{\mathcal{M}} A \text{ or } x \sqcup y \Vdash^-_{\mathcal{M}} B.$$

A formula $A \in \mathcal{F}_{\mathsf{LET}_{\mathsf{C}}}$ is verified (respectively, falsified) in $\mathcal{M}$ if $0 \Vdash^+_{\mathcal{M}} A$ (respectively, $0 \Vdash^-_{\mathcal{M}} A$).

Semilattice Semantics for $\mathsf{LET}_{\mathsf{C}}$

Similarly of what happens with Int, the completion of relational models for $\mathsf{LET}_{\mathsf{C}}$ does not necessarily yield valid relational models.

Semilattice Semantics for LET_C

Similarly of what happens with Int, the completion of relational models for LET_C does not necessarily yield valid relational models.

Definition 3.35

A tolerant relational model for LET_C is a structure $\langle S, \leqslant, v^+, v^- \rangle$, where $\langle S, \leqslant \rangle$ is a relational frame, $v^+ : V \cup \{\bot, \top\} \to \mathbf{P}(S)$ and $v^- : V \cup \{\bot, \top\} \to \mathbf{P}(S)$ are the verification and falsification valuation functions, respectively. These functions must satisfy the same conditions of valuations in semilattice models (Definition 3.34), with a slight modification to the last condition:

$$\text{If } x \in v^*(a), \text{then for all } y \in S, \text{ if } x \leqslant y, \text{ then } y \in v^*(a).$$

Semilattice Semantics for LET_C

Definition 3.36

Given a tolerant relational model $\mathcal{M} = \langle S, \leqslant, v^+, v^- \rangle$ for LET_C, the verification and falsification relations $\models^+_\mathcal{M}$ and $\models^-_\mathcal{M}$, between elements of S and formulas of LET_C, are simultaneously defined inductively by adding the following clauses to those given in Definition 3.25:

$$x \models^*_\mathcal{M} a \text{ iff } x \in v^*(a) \text{ (for } a \in \{\bot, \top\}\text{)};$$
$$x \models^+_\mathcal{M} B \prec A \text{ iff } x \models^+_\mathcal{M} B \text{ and } x \models^-_\mathcal{M} A;$$
$$x \models^-_\mathcal{M} B \prec A \text{ iff for all } y \in S, \text{ if } x \leqslant y, \text{ then } y \not\models^-_\mathcal{M} A \text{ or } y \models^-_\mathcal{M} B.$$

A formula $A \in \mathcal{F}_{\mathsf{LET}_\mathsf{C}}$ is verified (respectively, falsified) in $\mathcal{M}$, denoted by $\models^+_\mathcal{M} A$ (respectively, $\models^-_\mathcal{M} A$), if for all $x \in S$ it holds that $x \models^+_\mathcal{M} A$ (respectively, $x \models^-_\mathcal{M} A$).

Semilattice Semantics for LET_C

The adequacy of the tolerant relational semantics for LET_C is proved by means of the canonical model method.

Theorem 3.37 (Adequacy of the tolerant relational semantics for LET_C)

Let $\Gamma \cup \{A\} \subseteq \mathcal{F}_{\mathsf{LET}_\mathsf{C}}$. *We have that* $\Gamma \vdash_{\mathsf{LET}_\mathsf{C}} A$ *iff* $\Gamma \models_{\mathsf{LET}_\mathsf{C}} A$.

Semilattice Semantics for $\mathsf{LET_C}$

The adequacy of the tolerant relational semantics for $\mathsf{LET_C}$ is proved by means of the canonical model method.

Theorem 3.37 (Adequacy of the tolerant relational semantics for $\mathsf{LET_C}$)

Let $\Gamma \cup \{A\} \subseteq \mathcal{F}_{\mathsf{LET_C}}$. *We have that* $\Gamma \vdash_{\mathsf{LET_C}} A$ *iff* $\Gamma \models_{\mathsf{LET_C}} A$.

The equivalence between the tolerant relational semantics and the semilattice semantics for $\mathsf{LET_C}$ is proved by methods analogous to those used for the previous logics, thereby establishing the adequacy of the semilattice semantics for $\mathsf{LET_C}$.

Theorem 3.38 (Adequacy of the semilattice semantics for $\mathsf{LET_C}$)

Let $A \in \mathcal{F}_{\mathsf{LET_C}}$. *We have that* $\vdash_{\mathsf{LET_C}} A$ *iff* $\Vdash_{\mathsf{LET_C}} A$.

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Semilattice Semantics for $\mathsf{LET}_\jmath$

The extension of the semilattice semantics for BLE to $\mathsf{LET}_\jmath$ presents a different difficulty: in the relational semantics for $\mathsf{LET}_\jmath$, the valuation functions on propositional variables are not sufficient to completely determine the verifiability and falsifiability of all formulas.

Semilattice Semantics for $\mathrm{LET}_\jmath$

The extension of the semilattice semantics for BLE to $\mathrm{LET}_\jmath$ presents a different difficulty: in the relational semantics for $\mathrm{LET}_\jmath$, the valuation functions on propositional variables are not sufficient to completely determine the verifiability and falsifiability of all formulas.

Solution: Define verification and falsification valuation functions to be functions of the form $v^* : S \times \mathcal{F}_{\mathrm{LET}_\jmath} \to 0, 1$ that satisfy certain conditions. Intuitively, $v^+(x, A) = 1$ iff x supports the truth of A, and $v^-(x, A) = 1$ iff x supports the falsity of A.

Semilattice Semantics for LET_J

Definition 3.39

A semilattice model for LET_J is a structure $\langle S, \sqcup, 0, v^+, v^- \rangle$, where $\langle S, \sqcup, 0 \rangle$ is a semilattice frame, and $v^* : S \times \mathcal{F}_{\mathrm{LET}_\mathrm{J}} \to \{0, 1\}$ satisfy the following conditions:

$v^+(x, \sim A) = 1$ iff $v^-(x, A) = 1$;

$v^+(x, A \wedge B) = 1$ iff $v^+(x, A) = 1$ and $v^+(x, B) = 1$;

$v^+(x, A \vee B) = 1$ iff $v^+(x, A) = 1$ or $v^+(x, B) = 1$;

$v^+(x, A \to B) = 1$ iff for all $y \in S$ it holds that $v^+(y, A) = 0$ or $v^+(x \sqcup y, B) = 1$;

if $v^+(x, \circ A) = 1$, then for all $y \in S$ it holds that $v^+(x \sqcup y, A) = 1$ or $v^-(x \sqcup y, A) = 1$, and that $v^+(x \sqcup y, A) = v^-(x \sqcup y, A) = 1$ implies $v^+(x \sqcup y, B) = 1$;

$v^-(x, \sim A) = 1$ iff $v^+(x, A) = 1$;

$v^-(x, A \wedge B) = 1$ iff $v^-(x, A) = 1$ or $v^-(x, B) = 1$;

$v^-(x, A \vee B) = 1$ iff $v^-(x, A) = 1$ and $v^-(x, B) = 1$;

$v^-(x, A \to B) = 1$ iff $v^+(x, A) = 1$ and $v^-(x, B) = 1$;

if $v^*(x, A) = 1$, then for all $y \in S$ it holds that $v^*(x \sqcup y, A) = 1$.

Semilattice Semantics for $\mathsf{LET}_\jmath$

Definition 3.40

Given a semilattice model $\mathcal{M} = \langle S, \sqcup, 0, v^+, v^- \rangle$ for $\mathsf{LET}_\jmath$, the verification and falsification relations are define by $x \Vdash^*_{\mathcal{M}} A$ iff $v^*(x, A) = 1$.
A formula $A \in \mathcal{F}_{\mathsf{LET}_\jmath}$ is verified (respectively, falsified) in $\mathcal{M}$ if $0 \Vdash^+_{\mathcal{M}} A$ (respectively, $0 \Vdash^-_{\mathcal{M}} A$).

Semilattice Semantics for LET$_\text{J}$

Definition 3.41

A tolerant relational model for LET$_\text{J}$ is a structure $\langle S, \leqslant, v^+, v^- \rangle$, where $\langle S, \leqslant \rangle$ is a relational frame, and $v^+ : S \times \mathcal{F}_{\text{LET}_\text{J}} \to \{0,1\}$ and $v^+ : S \times \mathcal{F}_{\text{LET}_\text{J}} \to \{0,1\}$ are the verification and falsification valuation functions, respectively. They satisfy the same conditions as the valuation functions in Definition 3.39, suitably adapted in the clauses where the join operation is used, as follows:

$$v^+(x, A \to B) = 1 \text{ iff for all } y \in S, \text{ if } x \leqslant y, \text{ then } v^+(y, A) = 0 \text{ or } v^+(y, B) = 1;$$

$$\text{if } v^+(x, \circ A) = 1, \text{ then for all } y \in S, \text{ if } x \leqslant y, \text{ then } v^+(y, A) = 1 \text{ or } v^-(y, A) = 1,$$

$$\text{and } v^+(y, A) = v^-(y, A) = 1 \text{ implies } v^+(y, B) = 1;$$

$$\text{if } v^*(x, A) = 1, \text{ then for all } y \in S, \text{ if } x \leqslant y, \text{ then } v^*(y, A) = 1.$$

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Definition 3.42

Given a tolerant relational model $\mathcal{M} = \langle S, \leqslant, v^+, v^- \rangle$ for $\mathrm{LET}_\jmath$, the verification and falsification relations are define by $x \Vdash^*_{\mathcal{M}} A$ iff $v^*(x, A) = 1$.
A formula $A \in \mathcal{F}_{\mathrm{LET}_\jmath}$ is verified (respectively, falsified) in $\mathcal{M}$, denoted by $\models^+_{\mathcal{M}} A$ (respectively, $\models^-_{\mathcal{M}} A$), if for all $x \in S$ it holds that $x \models^+_{\mathcal{M}} A$ (respectively, $x \models^-_{\mathcal{M}} A$).

Semilattice Semantics for $\mathrm{LET}_\jmath$

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Semilattice Semantics for LET$_\jmath$

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Semilattice Semantics for LET$_\jmath$

- The adequacy of the tolerant relational semantics for LET$_\jmath$ is proved by using the canonical model method.
- The equivalence between the semilattice semantics and the tolerant relational semantics for LET$_\jmath$ is proved by methods similar to those used for the previous logics.
- The adequacy of the semilattice semantics for LET$_\jmath$ directly follows from the previous facts.

Highlighting the Role of Evidence in Deductions

- The valid formulas in the semilattice semantics for a logic L are those that are verified at the identity element of any semilattice model for L. Since the identity element is interpreted as the empty piece of information, the valid formulas are those that do not require any information to support their truth.

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- Considering information pieces as evidence, valid formulas are those whose truth can be accepted without requiring any evidence.
- To better understand the role of evidence, it is necessary to analyse how it operates in deductions from non-empty finite sets of hypotheses.

Highlighting the Role of Evidence in Deductions

Lemma 4.1

Let $\mathcal{M} = \langle S, \sqcup, 0, v^+, v^- \rangle$ *be a semilattice model for* L *and* $B_1, \ldots, B_n, A \in \mathcal{F}_{\mathsf{L}}$. *We have that* $0 \Vdash^+_{\mathcal{M}} B_1 \to (\ldots (B_n \to A) \ldots)$ *iff, for every* $x_1, \ldots, x_n \in S$, *if* $x_1 \Vdash^+_{\mathcal{M}} B_1$, $\ldots$, $x_n \Vdash^+_{\mathcal{M}} B_n$, *then* $x_1 \sqcup \ldots \sqcup x_n \Vdash^+_{\mathcal{M}} A$.

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Definition 4.2

Let $\mathcal{M} = \langle S, \sqcup, 0, v^+, v^- \rangle$ be a semilattice model for L and $B_1, \ldots, B_n, A \in \mathcal{F}_{\mathsf{L}}$, the formula A is a consequence of $\{B_1, \ldots, B_n\}$ in $\mathcal{M}$, denoted by $B_1, \ldots, B_n \Vdash_{\mathcal{M}} A$, if for every $x_1, \ldots, x_n \in S$, if $x_1 \Vdash^+_{\mathcal{M}} B_1, \ldots, x_n \Vdash^+_{\mathcal{M}} B_n$, then $x_1 \sqcup \ldots \sqcup x_n \Vdash^+_{\mathcal{M}} A$.

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Thus $B_1, \ldots, B_n \vdash_{\mathsf{L}} A$ iff $B_1, \ldots, B_n \Vdash_{\mathcal{M}} A$ for every semilattice model $\mathcal{M}$ for L. That is, A is a consequence of $\{B_1, \ldots, B_n\}$ in L iff, whenever there is evidence for accepting $B_1, \ldots, B_n$, the join of such evidence is evidence for accepting A.

Highlighting the Role of Evidence in Deductions

Although in the semilattice semantics the support of deductions by means of evidence is clear, this is not made explicit in the natural deduction system. We will add information sets to the natural deduction system in order to make the evidential support of deductions explicit.

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Although in the semilattice semantics the support of deductions by means of evidence is clear, this is not made explicit in the natural deduction system. We will add information sets to the natural deduction system in order to make the evidential support of deductions explicit.

Let us suppose a denumerable set of information pieces $I = \{i_j \mid j \in \mathbb{N}\}$. The elements of $\mathbf{P}_{\mathsf{Fin}}(I)$, the set of finite subsets of I, will be used to represent evidence. The letters N, M, L, ... will be used to denote arbitrary elements of $\mathbf{P}_{\mathsf{Fin}}(I)$, and expressions of the form $N : A$ will be used to represent the fact that evidence N supports the truth of formula A.

Highlighting the Role of Evidence in Deductions

Natural deduction rules with explicit evidence for BLE.

$$\frac{N : A \quad M : B}{N \cup M : A \wedge B}(I^e_\wedge)$$

$$\frac{N : A \wedge B}{N : A}(E1^e_\wedge) \qquad \frac{N : A \wedge B}{N : B}(E2^e_\wedge)$$

$$\frac{N : A}{N : A \vee B}(I1^e_\vee)$$

$$\frac{N : B}{N : A \vee B}(I2^e_\vee)$$

$$\frac{\begin{array}{c}[\{i_j\} : A] \\ \vdots \\ N : B\end{array}}{N \backslash \{i_j\} : A \to B}(I^e_\to)$$

$$\frac{\begin{array}{ccc}[\{i_j\} : A] & & [\{i_h\} : B] \\ \vdots & & \vdots \\ N : A \vee B & M : C & L : C\end{array}}{N \cup (M \backslash \{i_j\}) \cup (L \backslash \{i_h\}) : C}(E^e_\vee)$$

$$\frac{N : A \to B \quad M : A}{N \cup M : B}(E^e_\to)$$

Highlighting the Role of Evidence in Deductions

$$\frac{N : \sim A}{N : \sim(A \wedge B)}(I1^{e}_{\sim\,\wedge}) \qquad \frac{N : \sim B}{N : \sim(A \wedge B)}(I2^{e}_{\sim\,\wedge})$$

$$\frac{N : \sim A \quad M : \sim B}{N \cup M : \sim(A \vee B)}(I^{e}_{\sim\,\vee})$$

$$\frac{N : A \quad M : \sim B}{N \cup M : \sim(A \to B)}(I^{e}_{\sim\,\to})$$

$$\frac{N : A}{N : \sim\sim A}(I^{e}_{\sim\,\sim})$$

$$\frac{\begin{array}{ccc}[\{i_j\} : \sim A] & & [\{i_h\} : \sim B] \\ \vdots & & \vdots \\ N : \sim(A \wedge B) & M : C & L : C\end{array}}{N \cup (M \backslash \{i_j\}) \cup (L \backslash \{i_h\}) : C}(E^{e}_{\sim\,\wedge})$$

$$\frac{N : \sim(A \vee B)}{N : \sim A}(E1^{e}_{\sim\,\vee}) \qquad \frac{N : \sim(A \vee B)}{N : \sim B}(E2^{e}_{\sim\,\vee})$$

$$\frac{N : \sim(A \to B)}{N : A}(E1^{e}_{\sim\,\to}) \qquad \frac{N : \sim(A \to B)}{N : \sim B}(E2^{e}_{\sim\,\to})$$

$$\frac{N : \sim\sim A}{N : A}(E^{e}_{\sim\,\sim})$$

Highlighting the Role of Evidence in Deductions

Natural deduction rules with explicit evidence for $\prec$, $\bot$, and $\top$ in $\mathsf{LET_C}$.

$$\frac{N : \sim A \quad M : B}{N \cup M : B \prec A}(I^e_{\prec}) \qquad \frac{N : B \prec A}{N : \sim A}(E1^e_{\prec}) \quad \frac{N : B \prec A}{N : B}(E2^e_{\prec})$$

$$\frac{\begin{array}{c}[\{i_j\} : \sim A] \\ \vdots \\ N : \sim B\end{array}}{N \backslash \{i_j\} : \sim (B \prec A)}(I^e_{\sim \prec}) \qquad \frac{N : \sim (B \prec A) \quad M : \sim A}{N \cup M : \sim B}(E^e_{\sim \prec})$$

$$\frac{}{\varnothing : \top}(I^e_{\top}) \qquad \frac{N : \bot}{N : A}(E^e_{\bot})$$

$$\frac{}{\varnothing : \sim \bot}(I^e_{\sim \bot}) \qquad \frac{N : \sim \top}{N : A}(E^e_{\sim \top})$$

Highlighting the Role of Evidence in Deductions

Natural deduction rules with explicit evidence for $\circ$ in $\mathsf{LET_J}$.

$$\frac{N : \circ A \quad M : A \quad L : \sim A}{N \cup M \cup L : B}(EXP_\circ)$$

$$\frac{\begin{array}{ccc} & [\{i_j\} : A] & [\{i_h\} : \sim A] \\ & \vdots & \vdots \\ N : \circ A & M : B & L : B \end{array}}{N \cup (M \backslash \{i_j\}) \cup (L \backslash \{i_h\}) : B}(PEM_\circ)$$

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- A formula A is deducible from a set of formulas Γ in the natural deduction system with explicit evidence for L iff it is deducible in the corresponding natural deduction system without explicit evidence.

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Natural deduction rules with explicit evidence for $\circ$ in $\mathsf{LET_J}$.

$$\frac{N : \circ A \quad M : A \quad L : {\sim} A}{N \cup M \cup L : B}(EXP_\circ)$$

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- A formula A is deducible from a set of formulas Γ in the natural deduction system with explicit evidence for L iff it is deducible in the corresponding natural deduction system without explicit evidence.
- The system with explicit evidence makes it much clearer that deductions are supported by evidence and allows one to keep track of the pieces of information on which the conclusions are based.

Concluding Remarks and Future Work

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  - When semilattices are viewed as partial orders, semilattice models can be understood as relational models in which trivial states are allowed.

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- There is a close connection between relational semantics and semilattice semantics, at least for the logics considered here:
  - When semilattices are viewed as partial orders, semilattice models can be understood as relational models in which trivial states are allowed.
  - Any relational model can be extended to a semilattice model.
  - Whereas in relational models the elements of the partial order may be interpreted as information states, with the order relation representing extension of information, in semilattice semantics the elements of the semilattice represent pieces of information, and the join operation represents their combination.

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- In Weiss's semilattice semantics for Int (Weiss, 2021), the elements of the semilattice are interpreted as proofs. Under this interpretation, however, some elements are unintuitive:
  - The join operation is interpreted as the 'combination' of proofs, but such a notion is not clearly defined and, as Weiss himself acknowledges, the properties of such an operation, particularly commutativity, are open to objection.

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  - The identity element is interpreted as a 'null proof'; hence, any logical theorem is proved by a null proof.
- Semilattice semantics allows us to view the logics characterised by it, particularly the constructive logics considered here, as evidence-based logics.
- By incorporating semilattice terms into the natural deduction systems for BLE and LET_C, we have introduced a simple mechanism that makes the evidential support of deductions explicit and allows one to keep track of the pieces of information on which conclusions are based.

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- Possible future work:
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- Possible future work:
  - Extend semilattice semantics to other propositional and first-order logics.
  - There is currently a very active line of research connecting logic, formal reasoning, and explainable AI (XAI). The general idea is that logical methods can provide explanations that are transparent, formally verifiable, compositional, human-readable, and often causally meaningful. It would therefore be interesting to investigate whether the evidence-based perspective of BLE, $\mathsf{LET}_{\mathsf{C}}$, and $\mathsf{LET}_{\mathsf{J}}$ could contribute to the development of XAI models.

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