INTERACTION BETWEEN PROPERTIES OF AN IDEAL $\mathcal{I}$ ON $\mathbb{N}$ AND THE BANACH SPACE OF $\mathcal{I}$-NULL SEQUENCES

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An ideal on $\mathbb{N}$ is a collection $\mathcal{I}$ of subsets of $\mathbb{N}$ closed under finite unions and taking subsets of its elements. A sequence (x_n) in a Banach space X is $\mathcal{I}$-convergent to $x \in X$, and we write $\mathcal{I}\text{-}\lim x_n = x$, if $\{n \in \mathbb{N} : \|x_n - x\| \geq \varepsilon\} \in \mathcal{I}$ for each $\varepsilon > 0$. When $\mathcal{I}$ is Fin, the ideal of finite subsets of $\mathbb{N}$, this notion coincides with the classical convergence in X. The notion of $\mathcal{I}$-convergence was introduced in [7], although many authors had already studied this concept in particular cases and in different contexts (see, for instance, [1, 2, 3, 8]).

In this talk we present some results about the space

$$c_{0,\mathcal{I}}(X) = \{(x_n) \in \ell_\infty(X) : \mathcal{I}\text{-}\lim x_n = 0\}.$$

When X is the scalar field, we write $c_{0,\mathcal{I}}$ instead of $c_{0,\mathcal{I}}(X)$. The space $c_{0,\mathcal{I}}$ has recently received some attention (see, for instance, [6, 9]). It is known that $c_{0,\mathcal{I}}$ is a closed subspace (also, an ideal) of ℓ_∞, and that it is isometric to $C_0(U_\mathcal{I})$ for some open set $U_\mathcal{I}$ of $\beta\mathbb{N}$ (see [6]). Two extreme examples are worth keeping in mind. On the one hand, $c_{0,\mathsf{Fin}}$ is exactly c_0. On the other hand, if $\mathcal{I}$ is the trivial ideal $\mathcal{P}(\mathbb{N})$, then $c_{0,\mathcal{I}} = \ell_\infty$.

A natural question is to determine when two such spaces are isomorphic or isometric. As a consequence of the Banach–Stone theorem, $c_{0,\mathcal{I}}$ and $c_{0,\mathcal{J}}$ are isometric exactly when $\mathcal{I}$ and $\mathcal{J}$ are isomorphic as ideals [10]. However, the same does not hold for isomorphism.

Ideals on countable sets have been studied for a long time, and several ways of comparing them have been investigated (see the surveys [4, 12]). We are particularly interested in the Katětov pre-order (see, for instance, [5]). Given two ideals $\mathcal{I}$ and $\mathcal{J}$ on countable sets X and Y, respectively, we say that $\mathcal{I}$ is Katětov below $\mathcal{J}$, denoted $\mathcal{I} \leq_K \mathcal{J}$, if there exists a function $f : Y \to X$ such that $f^{-1}(A) \in \mathcal{J}$ for all $A \in \mathcal{I}$. We have shown in [10] that $\mathcal{I} \leq_K \mathcal{J}$ if and only if there exists a (not necessarily onto) Banach lattice isometry from $c_{0,\mathcal{I}}$ into $c_{0,\mathcal{J}}$ satisfying certain additional conditions.

More recently, we have shown [11] that the complementation of these spaces is related to a condition requiring that the ideal be the intersection of a countable family of maximal ideals.

This is joint work with Michael Rincón Villamizar.

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