▶ PAWEŁ GŁADKI AND KRZYSZTOF WORYTKIEWICZ, *On Locally o-minimal spaces.*
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In *Esquisse d'un programme* [2], Grothendieck formulated the concept of *tame topology* as a hypothetical notion of topology free from counter-intuitive results. O-minimal structures have been proposed as an axiomatic framework for this circle of ideas. Important results in real algebraic geometry have been achieved [5], yet the framework appears to be less compliant with algebraic topology. As observed by Delfs and Knebusch [1], semialgebraic coverings are necessarily of finite degree so in particular one does not get universal coverings. In fact, isomorphism classes of semialgebraic coverings of a semialgebraic space M over a real closed field R correspond to subgroups of finite index of the semialgebraic fundamental group $\pi_1^{\mathrm{semialg}}(M, x_0)$. It turns out that there are coverings classified by more general subgroups, yet they live in the category $\mathbf{LocSemialg}_R$ of *locally semialgebraic spaces*. The latter are possibly infinite gluings of semialgebraic patches. Coverings of arbitrary index including universal ones can be obtained that way. $\mathbf{LocSemialg}_R$ accomodates a fair amount of algebraic topology including various (co)homology theories as well as a homotopy theory. Delfs and Knebusch worked with the notion of *generalised topological spaces*, a presentation of sites of a certain type done in the spirit of point-set topology. Piękosz showed that the material carries over to the o-minimal context [4]. Still, the semialgebraic *stable homotopy theory* Delfs and Knebusch were looking for appears out of reach given the deployed techniques and this is the starting point of this work. We reformulate the notion of generalised topological spaces from first principles. A locally o-minimal space is then a lattice of unions of open definable sets equipped with a subcanonical coverage including all finite covers. A formal similarity with the category of schemes becomes apparent at this point, allowing to channel the quest for a stable o-minimal homotopy theory along the lines of motivic homotopy theory [6]. We characterise the category of locally o-minimal spaces as a certain subcategory of $\mathbf{Shv}\left(\mathsf{Aff}_{\mathrm{R}}, \tau\right)$, the topos of sheaves over the site $\mathsf{Aff}_{\mathrm{R}}$ of unions of definable open subsets of a definable open set. Furthermore, we set up an *interval-based* model structure on $\mathbf{Shv}\left(\mathsf{Aff}_{\mathrm{R}}, \tau\right)$ [3] and identify the salient features of the homotopy theory it produces. This homotopy theory is not stable, yet we expect that the stable homotopy theory of locally o-minimal spaces lives in a category of spectra over $\mathbf{Shv}\left(\mathsf{Aff}_{\mathrm{R}}, \tau\right)$.

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[2] Alexandre Grothendieck, ***Esquisse d'un programme***, Chez l'auteur, 1984.

[3] John F Jardine, ***Local homotopy theory***, Springer, 2015.

[4] Artur Piękosz, *O-minimal homotopy and generalized (co) homology*, ***The Rocky Mountain Journal of Mathematics***, vol. 1 (2013), pp. 573–617.

[5] Lou Van Den Dries, *O-minimal structures and real analytic geometry*, ***Current developments in mathematics***, vol. 1 (1998), no. X, pp. 105–152.

[6] Vladimir Voevodsky, *A1-homotopy theory*, ***Proceedings of the international congress of mathematicians*** (Berlin), (First Last and First2 Last2, editors), vol. 1, 1998, pp. 579–604.